Dynamic Systems and Applications 18 (2009) 303-322

NONLINEAR QUADRATIC FIRST ORDER FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

B. C. DHAGE

Kasubai, Gurukul Colony, Almedpur 413515, Maharashtra, India

ABSTRACT. In this paper, an existence theorem for the periodic boundary value problems of first order quadratic functional integro-differential equations is proved via a fixed point theorem in Banach algebras and under some mixed generalized Lipschitz and Carathéodory conditions. The existence theorems for extremal positive solutions are also proved under certain monotonicity conditions.

AMS (MOS) Subject Classification. 34K10

1. INTRODUCTION

First order ordinary functional differential equations (ODE) with periodic boundary value conditions are considered in many works. See Bernfeld and Lakshmikantham [1], Ladde *et al.* [17], Omari and Zanolin [20] and the references therein. The study of periodic boundary value problems of nonlinear first order functional differential equations with discontinuous nonlinearity has been exploited in the works of Heikkilä and Lakshmikantham [16]. But the study of periodic boundary value problems of quadratic ordinary functional integro-differential equations involving Carathódory as well as discontinuous nonlinearity has not been made so far in the literature. The study of initial value problems of nonlinear quadratic functional differential and integral equations is initiated in the works of Dhage [2] and Dhage and O'Regan [10] and discussed the existence theory for first order functional differential and integral equations. The study of such equations has been further exploited in the works of Dhage [3, 4, 5, 7] and Dhage *et al.* [11] for various aspects of the solutions. In this paper, we deal with the periodic boundary value problems of nonlinear first order quadratic functional differential equations and discuss the existence as well as existence results for extremal solutions under mixed Lipschitz, Carathéodory and monotonic conditions. The main tools used in the study are the hybrid fixed point theorems of Dhage [3, 4, 6, 7]. We claim that the nonlinear functional equation as well as the existence results of this paper are new to the literature on the theory of nonlinear ordinary functional equations.

Received May 1, 2008

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the periodic boundary value problems (in short PBVP) of first order ordinary functional integro-differential equations

$$\left.\begin{aligned} \frac{d}{dt}\Big[\frac{x(t)}{f(t, x(t), x(\mu(t)))}\Big] &= g\Big(t, x(\theta(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(t)))\, ds\Big) \quad \text{a.e. } t \in J \\ x(0) &= x(T), \end{aligned}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mu, \theta, \sigma, \eta : J \to J$.

By a *solution* of the PBVP (1.1) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \Big(\dfrac{x(t)}{f(t, x(t), x(\mu(t)))}\Big)$ is absolutely continuous on J, and
(ii) x satisfies the equations in (1.1),

where $AC(J, \mathbb{R})$ is the space of continuous functions whose first derivative exists and is absolutely continuous real-valued functions on J.

The periodic boundary value problem (1.1) is quite general in the sense that it includes several known classes of periodic boundary value problems as special cases. For example, if $f(t, x, y) = 1$ on $J \times \mathbb{R} \times \mathbb{R}$, then PBVP (1.1) reduces to the PBVP

$$\left.\begin{aligned} x'(t) &= g\Big(t, x(\theta(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(t)))\, ds\Big) \quad \text{a. e. } t \in J \\ x(0) &= x(T), \end{aligned}\right\} \tag{1.2}$$

which further, when $g(t, x, y) = g(t, x)$ on$J \times \mathbb{R} \times \mathbb{R}$, and θ is identity map on J, includes the following PBVP studied in Nieto [18, 19],

$$\left.\begin{aligned} x'(t) &= g(t, x(t)) \quad \text{a. e. } t \in J \\ x(0) &= x(T). \end{aligned}\right\} \tag{1.3}$$

There is good deal of literature on the PBVP (1.3) for different aspects of the solutions. In this paper,we discuss the PBVP (1.1) for existence theory only under suitable conditions on the nonlinearities f and g involved in it.

Our method of study is to convert the PBVP (1.1) into an equivalent integral equation and apply the fixed point theorems of Dhage [3, 4, 6, 7] under suitable conditions on the nonlinearities f and g involved in it. In the following section 2, we prove the main existence theorem and the existence theorems for extremal solutions are presented in section 3. Finally, an illustrative example is given at the end of the paper.

2. **EXISTENCE THEORY**

Let $B(J, \mathbb{R})$ denote the space of bounded real-valued functions on J. Let $C(J, \mathbb{R})$, denote the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ and a multiplication " · " in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \quad \text{and} \quad (x.y)(t) = x(t)y(t) \ \text{ for } \ t \in J.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with respect to above norm and multiplication. By $L^1(J,\mathbb{R})$ we denote the set of Lebesgue integrable functions on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,ds.$$

We employ a hybrid fixed point theorem of Dhage [7] for proving the existence result for the IVP (1.1). Before stating this fixed point theorem, we give some preliminaries.

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called **$\mathcal{D}$-Lipschitz** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ $(\alpha > 0)$, A is called a **Lipschitz** with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a **nonlinear $\mathcal{D}$-contraction** on X. Sometimes we call the function ψ a **$\mathcal{D}$-function** for convenience.

An operator $T : X \to X$ is called **compact** if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly $T : X \to X$ is called **totally bounded** if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called **completely continuous** operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [7] is embodied in the following theorem. Also see Dhage and Ntouyas [8], Dhage *et al.* [9] and the references therein.

Theorem 2.1 (Dhage [7]). *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively open and closed balls in a Banach algebra X centered at origin 0 and of radius r. Let $A, B : \overline{\mathcal{B}_r(0)} \to X$ be two operators satisfying*

(a) *A is Lipschitz with the Lipschitz constant α,*
(b) *B is compact and continuous, and*
(c) *$\alpha M < 1$, where $M = \|B(\overline{\mathcal{B}_r(0)})\| := \sup\{\|Bx\| : x \in \overline{\mathcal{B}_r(0)}\}$.*

Then either

(i) *the equation $\lambda[Ax\,Bx] = x$ has a solution for $\lambda = 1$, or*
(ii) *there exists an $u \in X$ such that $\|u\| = r$ satisfying $\lambda[Au\,Bu] = u$ for some $0 < \lambda < 1$.*

The following useful lemma is obvious and the details may be found in Nieto [19].

Lemma 2.2. *For any* $h \in L^1(J, \mathbb{R}^+)$ *and* $\sigma \in L^1(J, \mathbb{R})$, x *is a solution to the functional equation*

$$\left.\begin{aligned} x' + h(t)x(t) = \sigma(t) \ \ a.\ \ e.\ t \in J \\ x(0) = x(T), \end{aligned}\right\} \tag{2.2}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t,s)\sigma(s)\,ds \tag{2.3}$$

where

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)}}{1-e^{-H(T)}}, & 0 \le s \le t \le T, \\ \\ \dfrac{e^{H(s)-H(t)-H(T)}}{1-e^{-H(T)}}, & 0 \le t < s \le T, \end{cases} \tag{2.4}$$

where $H(t) = \int_0^t h(s)\,ds$.

Notice that the Green's function G_k is nonnegative on $J \times J$ and the number

$$M_h := \max\{\,|G_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J, \mathbb{R}^+)$. Note also that $H(t) > 0$ for all $t > 0$.

We need the following definitions in the sequel.

Definition 2.3. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathcal{D}$**-function** if it satisfies

(*i*) ψ is continuous,
(*ii*) ψ is nondecreasing , and
(*iii*) ψ is scalarly submultiplicative, that is, $\psi(\lambda r) \le \lambda\psi(r)$ for all $\lambda > 0$ and $r \in \mathbb{R}^+$.

The class of all $\mathcal{D}$-functions on $\mathbb{R}^+$ is denoted by Ψ. There do exist $\mathcal{D}$-functions on $\mathbb{R}$. Indeed, the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\psi(r) = \ell r,\ \ell > 0$ satisfies the conditions $(i) - (iii)$ mentioned above and hence a $\mathcal{D}$-function on $\mathbb{R}^+$. Note that if $\psi \in \Psi$ then $\psi(0) = 0$.

Definition 2.4. A mapping $\beta : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be **Carathéodory** if

(i) $t \mapsto \beta(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$, and
(ii) $(x,y) \mapsto \beta(t,x,y)$ is continuous almost everywhere for $t \in J$.

Again a Carathéodory function $\beta(t,x,y)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $q_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \le q_r(t), \quad a.e. \ \ t \in J$$

for all $x, y \in \mathbb{R}$ with $|x| \le r$ and $|y| \le r$.

Finally a Carathéodory function $\beta(t, x, y)$ is called L^1_X-Carathéodory if

(iv) there exists a function $q \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \le q(t), \quad a.e. \ \ t \in J$$

for all $x, y \in \mathbb{R}$.

For convenience, the function q is referred to as a **bound function** of β.

We will use the following hypotheses in the sequel.

(A_0) The functions θ, $\eta : J \to J$ are measurable and the functions μ, $\sigma : J \to J$ are continuous with $\mu(0) = 0$ and $\mu(T) = T$.

(A_1) The function $t \mapsto f(t, x, y)$ is periodic of period T for all $x, y \in \mathbb{R}$.

(A_2) The function $x \mapsto \dfrac{x}{f(0, x, x)}$ is injective in $\mathbb{R}$.

(A_3) The function f is continuous on $J \times \mathbb{R} \times \mathbb{R}$ and there exists a bounded function $l : J \to \mathbb{R}^+$ with bound L such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le l(t) \max\{|x_1 - y_1|\,, |x_2 - y_2|\} \ \text{ a.e.} \quad t \in J$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(A_4) The function k is continuous on $J \times \mathbb{R} \times \mathbb{R}$ and there exists a function $\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$|k(t, s, x)| \le \alpha(s)|x|$$

for all $t, s \in J$ and $x \in \mathbb{R}$.

(A_5) The function g is Carathéodory on $J \times \mathbb{R} \times \mathbb{R}$.

Note that hypotheses (A_0) through (A_3) are much common in the literature on the theory of nonlinear functional equations. Actually the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(t, x, y) = \alpha + \beta(x + y)$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta(x + y) \neq 0$ satisfies the hypotheses (A_0)-(A_3).

Now consider the PBVP

$$\left.\begin{aligned} &\Big(\frac{x(t)}{f(t, x(t), x(\mu(t)))}\Big)' + h(t)\Big(\frac{x(t)}{f(t, x(t), x(\mu(t)))}\Big) \\ &\qquad\qquad = g_h\Big(t, x(\theta(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(t)))\, ds\Big) \ \text{a.e. } t \in J \\ &x(0) = x(T) \end{aligned}\right\} \tag{2.5}$$

where $h \in L^1(J,\mathbb{R}^+)$ is bounded and the function $g_h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g_h(t,x,y) = g(t,x,y) + h(t)\Big(\frac{x}{f(t,x,x(\mu))}\Big). \tag{2.6}$$

Remark 2.5. Note that the PBVP (1.1) is equivalent to the PBVP (2.5) and a solution of the PBVP (1.1) is the solution for the PBVP (2.5) on J and vice versa.

Remark 2.6. Assume that hypotheses (A_2) and (A_4) hold. Then the function g_h defined by (2.5) is Carathéodory on $J \times \mathbb{R} \times \mathbb{R}$.

Lemma 2.7. *Assume that hypotheses (A_0)-(A_2) hold. Then for any bounded $h \in L^1(J,\mathbb{R}^+)$, x is a solution to the functional equation (2.5) if and only if it is a solution of the integral equation*

$$x(t) = \big[f(t,x(t),x(\mu(t)))\big]\Big(\int_0^T G_h(t,s)g_h\Big(s,x(\theta(s)),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds\Big) \tag{2.7}$$

for all $t \in J$, where the Green's function $G_h(t,s)$ is defined by (2.4).

Proof. Let $y(t) = \dfrac{x(t)}{f(t,x(t),x(\mu(t)))}$. Since $f(t,x,y)$ is periodic in t of period T for all $x,y \in \mathbb{R}$, we have

$$y(0) = \frac{x(0)}{f(0,x(0),x(0))} = \frac{x(T)}{f(T,x(T),x(T))} = y(T).$$

Now an application of Lemma 2.2 yields that the solution to functional differential equation (2.5) is the solution to integral equation (2.7). Conversely, suppose that x is any solution to the integral equation (2.7), then

$$y(0) = \frac{x(0)}{f(0,x(0),x(0))} = y(T) = \frac{x(T)}{f(T,x(T),x(T))} = \frac{x(T)}{f(0,x(T),x(T))}.$$

Since the function $x \mapsto \dfrac{x}{f(0,x,x)}$ is injective, one has $x(0) = x(T)$ and so, x is a solution to PBVP (1.1). The proof of the lemma is complete. □

We make use of the following hypothesis in the sequel.

(A_6) There exists a function $\gamma \in L^1(J,\mathbb{R}^+)$ and a $\mathcal{D}$-function $\psi \in \Psi$ such that

$$|g_h(t,x,y)| \le \gamma(t)\psi(|x|+|y|) \quad \text{a.e. } t \in J \tag{2.8}$$

whenever $x,y \in \mathbb{R}$.

We frequently make use of the following estimate concerning the function $g(t,x,y)$ in the sequel.

If the hypotheses (A_4)-(A_6) hold, then for any $x \in C(J,\mathbb{R})$ with $\|x\| \le r$, one has

$$
\begin{aligned}
\Big| g\Big(t, x(\theta(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\Big| & \\
&\le \gamma(t)\psi\Big(|x(\theta(t))| + \int_0^{\sigma(t)} |k(t,s,x(\eta(s)))|\,ds\Big)\\
&\le \gamma(t)\psi\Big(\|x\| + \int_0^{\sigma(t)} \alpha(s)|x(\eta(s))|\,ds\Big)\\
&\le \gamma(t)\psi\Big(\|x\| + \int_0^{T} \alpha(s)\|x\|\,ds\Big)\\
&\le \gamma(t)\psi\Big([1+\|\alpha\|_{L^1}]\,\|x\|\Big)\\
&\le \gamma(t)(1+\|\alpha\|_{L^1})\psi(r) \qquad (2.9)
\end{aligned}
$$

for all $t \in J$.

Theorem 2.8. *Assume that the hypotheses (A_0)-(A_1), (A_3)–(A_6) hold. Suppose that there exists a real number $r > 0$ such that*

$$
r > \frac{FM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)}{1 - LM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)} \qquad (2.10)
$$

where, $LM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r) < 1$, $F = \sup_{t\in[0,T]} |f(t,0,0)|$ and $L = \max_{t\in J} \ell(t)$. Then the PBVP (1.1) has a solution on J.

Proof. Let $X = C(J,\mathbb{R})$. Define an open ball $\mathcal{B}_r(0)$ centered at origin 0 of radius r, where the real number r satisfies the inequality (2.10). Define two mappings A and B on $\overline{\mathcal{B}_r(0)}$ by

$$
Ax(t) = f(t, x(t), x(\mu(t))), \quad t \in J, \qquad (2.11)
$$

and

$$
Bx(t) = \int_0^T G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds, \quad t \in J. \qquad (2.12)
$$

Obviously, A and B define the operators $A, B : \overline{\mathcal{B}_r(0)} \to X$. Then the integral equation (2.7) is equivalent to the operator equation

$$
Ax(t)\,Bx(t) = x(t), \quad t \in J. \qquad (2.13)
$$

We shall show that the operators A and B satisfy all the hypotheses of Theorem 2.1. We first show that A is a Lipschitz on X. Let $x, y \in X$. Then by (A_3),

$$
\begin{aligned}
|Ax(t) - Ay(t)| &= |f(t,x(t),x(\mu(t))) - f(t,y(t),y(\mu(t)))|\\
&\le \ell(t)\max\{|x(t)-y(t)|\,,\,|x(\mu(t)) - y(\mu(t))|\}\\
&\le L\,\|x-y\|
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \le L\|x - y\|$$

for all $x, y \in X$. So A is a Lipschitz on X with Lipschitz constant L. Next we show that B is completely continuous on X. Using the standard arguments as in Granas *et al.* [14], it is shown that B is a continuous operator on X. We shall show that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equicontinuous set in X. Let $x \in \overline{\mathcal{B}_r(0)}$ be arbitrary. Since g is Carathéodory, we have

$$\begin{aligned}
|Bx(t)| &\le \Big| \int_0^T G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds\Big| \\
&\le \int_0^T \Big| G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\Big|\,ds \\
&\le \int_0^T G_h(t,s)\gamma(t)\psi\Big(\|x\| + \int_0^{\sigma(s)} \alpha(\tau)|x(\eta(\tau))|\,d\tau\Big)ds \\
&\le M_h \int_0^T \gamma(s)\psi\Big(\|x\| + \int_0^T \alpha(\tau)\|x\|\,d\tau\Big)ds \\
&\le M_h \int_0^T \gamma(s)(1 + \|\alpha\|_{L^1})\psi(r)\,ds \\
&\le M_h \|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r).
\end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \le M$ for all $x \in \overline{\mathcal{B}_r(0)}$, where $M = M_h\|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r)$ This shows that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded set in X. Next we show that $B(\overline{\mathcal{B}_r(0)})$ is an equicontinuous set. To finish it is enough to show that $y' = (Bx)'$ is bounded on $[0, T]$. Now for any $t \in [0, T]$, one has

$$\begin{aligned}
|y'(t)| &\le \Big| \int_0^T \frac{\partial}{\partial t} G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds\Big| \\
&= \Big| \int_0^T |(-h(t))| G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds\Big| \\
&\le H\, M_h \|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r) \\
&= c,
\end{aligned}$$

where $H = \max_{t\in J} h(t)$. Hence for any $t, \tau \in [0, T]$ one has

$$|Bx(t) - Bx(\tau)| \le c\,|t - \tau| \to 0 \quad \text{as} \quad t \to \tau$$

uniformly for all $x \in \overline{\mathcal{B}_r(0)}$. This shows that $B(\overline{\mathcal{B}_r(0)})$ is a equi-continuous set X. Now $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equi-continuous set in X, so it is compact by Arzelà-Ascoli theorem. As a result B is a compact and continuous operator on $\overline{\mathcal{B}_r(0)}$. Finally, by hypothesis,

$$\alpha M = L M_h \|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r) < 1,$$

and thus all the conditions of Theorem 2.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be a solution to PBVP (1.1) such that $\|u\| = r$. Then we have, for any $\lambda \in (0,1)$,

$$u(t) = \lambda \left[f(t, u(t), u(\mu(t))) \right] \left(\int_0^T G_h(t,s) g_h\Big(s, u(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau)))\,d\tau\Big)\,ds \right)$$

for $t \in J$. Therefore,

$$\begin{aligned}
|u(t)| &\le \lambda\, |f(t,u(t),u(\mu(t)))| \\
&\quad \times \left(\left| \int_0^T G_h(t,s) g_h\Big(s, u(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau)))\,d\tau\Big)\,ds \right| \right) \\
&\le \lambda \Big(\big| f(t,u(t),u(\mu(t))) - f(t,0,0) \big| + \big| f(t,0,0) \big| \Big) \\
&\quad \times \left(\int_0^T G_h(t,s) \Big| g_h\Big(s, u(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau)))\,d\tau\Big) \Big|\,ds \right) \\
&\le \big[\ell(t)\, \max\{|u(t)|, |u(\mu(t))|\} + F \big] \\
&\quad \times \left(\int_0^T M_h \Big| g_h\Big(s, u(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau)))\,d\tau\Big) \Big|\,ds \right) \\
&\le L M_h \, \|u\| \left(\int_0^T \gamma(s)(1 + \|\alpha\|_{L^1}) \psi(\|u\|)\,ds \right) \\
&\quad + F M_h \left(\int_0^T \gamma(s)(1 + \|\alpha\|_{L^1}) \psi(\|u\|)\,ds \right) \\
&\le L M_h \|\gamma\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(\|u\|) \|u\| + F M_h \|\gamma\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(\|u\|). \qquad (2.14)
\end{aligned}$$

Taking the supremum in the above inequality (2.11) yields

$$\|u\| \le \frac{F M_h \|\gamma\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(\|u\|)}{1 - L M_h \|\gamma\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(\|u\|)}.$$

Substituting $\|u\| = r$ in above inequality yields

$$r \le \frac{F M_h \|\gamma\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(r)}{1 - L M_h \|\gamma\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(r)}.$$

This is a contradiction to (2.10). Hence the conclusion (ii) of Theorem 2.1 does not hold. Therefore the operator equation $Ax\,Bx = x$ and consequently the PBVP (1.1) has a solution on J. This completes the proof. □

Remark 2.9. We note that in Theorem 2.8, we only require the hypothesis (A_2) to hold in $[-r, r]$.

3. **EXISTENCE OF EXTREMAL SOLUTIONS**

A non-empty closed set K in a Banach algebra X is called a **cone** if (i) $K+K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called to be **positive** if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called to be **normal** if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [15].

Lemma 3.1 (Dhage [4])**.** *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1u_2 \leq v_1v_2$.*

For any $a, b \in X, a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

Definition 3.2. A mapping $T : [a, b] \to X$ is said to be **nondecreasing** or **monotone increasing** if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in [a, b]$.

We equip the space $C(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}. \tag{3.1}$$

It is well known that the cone K is positive and normal in $C(J, \mathbb{R})$. We need the following definitions in the sequel.

Definition 3.3. A function $a \in AC(J, \mathbb{R})$ is called a lower solution of the PBVP (1.1) on J if the function $t \mapsto \Big(\dfrac{a(t)}{f(t, a(t), a(\mu(t)))}\Big)$ is absolutely continuous on J and

$$\left.\begin{aligned} \frac{d}{dt}\Big[\frac{a(t)}{f(t, a(t), a(\mu(t)))}\Big] &\leq g\Big(t, a(\theta(t)), \int_0^{\sigma(t)} k(t, s, a(\eta(s)))\, ds\Big) \text{ a.e. } t \in J \\ a(0) &\leq a(T). \end{aligned}\right\}$$

Similarly, a function $b \in AC(J, \mathbb{R})$ is called an upper solution of the PBVP (1.1) on J if the function $t \mapsto \Big(\dfrac{b(t)}{f(t, b(t), b(\mu(t)))}\Big)$ is absolutely continuous on J and

$$\left.\begin{aligned} \frac{d}{dt}\Big[\frac{b(t)}{f(t, b(t), b(\mu(t)))}\Big] &\geq g\Big(t, b(\theta(t)), \int_0^{\sigma(t)} k(t, s, b(\eta(s)))\, ds\Big) \text{ a.e. } t \in J \\ b(0) &\geq b(T). \end{aligned}\right\}$$

Definition 3.4. A solution x_M of the PBVP (1.1) is said to be maximal if for any other solution x to PBVP (1.1) one has $x(t) \leq x_M(t)$, for all $t \in J$. Again a solution x_m of the PBVP (1.1) is said to be minimal if $x_m(t) \leq x(t)$, for all $t \in J$, where x is any solution of the PBVP (1.1) on J.

Remark 3.5. The upper and lower solutions of the PBVP (1.1) are respectively the upper and lower solutions of the PBVP (2.5) and vice-versa. Similarly the maximal and minimal solutions of the PBVP (1.1) are respectively the upper and lower solutions of the PBVP (2.5) and vice-versa.

3.1. **Carathéodory case.** We use the following fixed point theorems of Dhage [3] for proving the existence of extremal solutions for the BVP (1.1) under certain monotonicity conditions.

Theorem 3.6 (Dhage [3])**.** *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two nondecreasing operators such that*

(a) *A is Lipschitz with the Lipschitz constant α,*
(b) *B is completely continuous, and*
(c) *$Ax\, Bx \in [a, b]$ for each $x \in [a, b]$.*

Further if the cone K is positive and normal, then the operator equation $Ax\, Bx = x$ has the least and the greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Remark 3.7. Note that hypothesis (c) of Theorem 3.6 holds if the operators A and B are positive, monotone increasing and there exist elements a and b in X such that $a \leq Aa\, Ba$ and $Ab\, Bb \leq b$.

We need the following definition in the sequel.

Definition 3.8. A function $f : \mathbb{R} \to \mathbb{R}$ is called nondecreasing if $f(x) \leq f(y)$ for all $x, y \in \mathbb{R}$ with $x \leq y$. Similarly, f is called increasing in x if $f(x) < f(y)$ for all $x, y \in \mathbb{R}$ with $x < y$.

We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g_h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$.
(B_1) The function $x \mapsto \dfrac{x}{f(0, x, x)}$ is increasing in the interval $\big[\min_{t \in J} a(t), \max_{t \in J} b(t)\big]$.
(B_2) The function $k(t, s, x)$ is nondecreasing in x for $t, s \in J$.
(B_3) The functions $f(t, x, y)$ and $g_h(t, x, y)$ are nondecreasing in x and y almost everywhere for $t \in J$.
(B_4) The PBVP (1.1) has a lower solution a and an upper solution b on J with $a \leq b$.

(B_5) The function $q : J \to \mathbb{R}$ defined by

$$q(t) = g_h\Big(t, b(\theta(t)), \int_0^{\sigma(t)} k(t, s, b(\eta(t)))\, ds\Big)$$

is Lebesgue integrable.

We remark that hypothesis (B_5) holds in particular g is L^1-Carathéodory on $J \times \mathbb{R} \times \mathbb{R}$.

Remark 3.9. If the hypotheses (B_1) -(B_4) holds, then the map $x \mapsto \dfrac{x}{f(0,x,x)}$ is injective and

$$\frac{a(0)}{f(0, a(0), a(0))} \le \frac{a(T)}{f(T, a(T), a(T))} \quad \text{and} \quad \frac{b(0)}{f(0, b(0), b(0))} \ge \frac{b(T)}{f(T, b(T), b(T))}$$

which guarantee that $a \le Aa\, Ba$ and $Ab\, Ba \le b$.

Remark 3.10. Assume that hypotheses (B_0) through (B_5) hold. Then the function $t \mapsto g_h\Big(t, x(\theta(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s)))\, ds\Big)$ is Lebesgue integrable on J and

$$\Big|g_h\Big(t, x(\theta(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s)))\, ds\Big)\Big| \le q(t), \quad \text{a.e. } t \in J,$$

for all $x \in [a, b]$.

Theorem 3.11. *Suppose that the assumptions (A_0)-(A_1), (A_3), (A_5) and (B_0)-(B_5) hold. Furthermore, if $LT\|q\|_{L^1} < 1$, where $L = \max_{t\in J}\, \ell(t)$, then PBVP (1.1) has a minimal and a maximal positive solution defined on J.*

Proof. Now PBVP (1.1) is equivalent to integral equation (2.7) on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on X by (2.11) and (2.12) respectively. Then integral equation (2.7) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 2.8, that A is a Lipschitz with a Lipschitz constant L and B is completely continuous operator on $[a, b]$. Again the hypothesis (B_2)-(B_3) implies that A and B are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \le y$. Then by (B_3),

$$Ax(t) = f(t, x(t), x(\mu(t))) \le f(t, y(t), y(\mu(t))) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned} Bx(t) &= \int_0^T G_h(t, s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s, \tau, x(\eta(\tau)))\, d\tau\Big)\, ds \\ &\le \int_0^T G_h(t, s) g_h\Big(s, y(\theta(s)), \int_0^{\sigma(s)} k(s, \tau, y(\eta(\tau)))\, d\tau\Big)\, ds \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[a, b]$. Again Lemma 2.1, Remark 3.9 and hypothesis (B_4) together imply that

$$\begin{aligned}
a(t) &\le [f(t, a(t), a(\mu(t)))] \left(\int_0^T G_h(t,s) g_h\Big(s, a(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,a(\eta(\tau)))\,d\tau\Big)\,ds \right) \\
&\le [f(t, x(t), x(\mu(t)))] \\
&\quad \times \left(\int_0^T G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds \right) \\
&\le [f(t, b(t), b(\mu(t)))] \left(\int_0^T G_h(t,s) g_h\Big(s, b(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,b(\eta(\tau)))\,d\tau\Big)\,ds \right) \\
&\le b(t),
\end{aligned}$$

for all $t \in J$ and $x \in [a, b]$. As a result $a(t) \le Ax(t)Bx(t) \le b(t)$, for all $t \in J$ and $x \in [a, b]$. Hence $Ax\,Bx \in [a, b]$ for all $x \in [a, b]$. Again,

$$\begin{aligned}
M &= \|B([a,b])\| \\
&= \sup\{\|Bx\| : x \in [a,b]\} \\
&\le \sup\left\{ \sup_{t\in J} \int_0^T G_h(t,s) \Big| g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big) \Big|\,ds \mid x \in [a,b] \right\} \\
&\le M_h \int_0^T q(s)\,ds \\
&= M_h \|q\|_{L^1}.
\end{aligned}$$

Since $\alpha M \le L M_h \|q\|_{L^1} < 1$, we apply Theorem 3.6 to the operator equation $Ax\,Bx = x$ to yield that the PBVP (1.1) has a minimal and a maximal positive solution defined on J. This completes the proof. □

3.2. **Discontinuous case.** We use the following fixed point theorems of Dhage [6] for proving the existence of extremal solutions for the BVP (1.1) when the right hand side function g is discontinuous on $J \times \mathbb{R} \times \mathbb{R}$.

Theorem 3.12 (Dhage [6])**.** *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two nondecreasing operators such that*

(a) *A is completely continuous,*
(b) *B is totally bounded, and*
(c) *$Ax\,By \in [a, b]$ for each $x, y \in [a, b]$.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has the least and the greatest positive solution in $[a, b]$.

Theorem 3.13. (Dhage [6]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two nondecreasing operators such that*

(a) *A is Lipschitz with the Lipschitz constant α,*
(b) *B is totally bounded, and*
(c) *$Ax\, By \in [a, b]$ for each $x, y \in [a, b]$.*

Further if the cone K is positive and normal, then the operator equation $Ax\, Bx = x$ has the least and the greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Remark 3.14. Note that hypothesis (c) of Theorems 3.12, and 3.13 holds if the operators A and B are positive, monotone increasing and there exist elements a and b in X such that $a \leq Aa\, Ba$ and $Ab\, Bb \leq b$.

We need the following definition in the sequel.

Definition 3.15. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be **Chandrabhan** if

(i) $t \mapsto \beta(t, x, y)$ is measurable for each $x, y \in C(J, \mathbb{R})$, and
(ii) $\beta(t, x, y)$ is nondecreasing in x and y almost everywhere for $t \in J$.

Again a Chandrabhan function $\beta(t, x, y)$ is called L^1-Chandrabhan if

(iii) for each real number $r > 0$ there exists a function $q_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \leq q_r(t), \quad a.e.\ t \in J$$

for all $x, y \in \mathbb{R}$ with $|x| \leq r$ and $|y| \leq r$.

Finally a Chandrabhan function $\beta(t, x, y)$ is called L^1_X-Chandrabhan if

(iv) there exists a function $q \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \leq q(t), \quad a.e.\ t \in I$$

for all $x, y \in \mathbb{R}$.

For convenience, the function h is referred to as a **bound function** of β.

We consider the following hypotheses in the sequel.

(C_1) The function $f(t, x, y)$ is nondecreasing in x and y almost everywhere for $t \in J$.
(C_2) The function g_h defined by (2.6) is Chandrabhan.

Theorem 3.16. *Suppose that the assumptions (A_0)-(A_1), (B_0)-(B_2), (B_4)-(B_5)and (C_1)-(C_2) hold. Then PBVP (1.1) has a minimal and a maximal positive solution defined on J.*

Proof. Now PBVP (1.1) is equivalent to integral equation (2.7) on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on X by (2.11) and (2.12) respectively. Then integral equation (2.7) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [a, b] \to K$. Note that the conditions (B_1) and (B_1) provides $a \leq Aa\,Ba$ and $Ab\,Bb \leq b$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X.

Step I : First we show that A is completely continuous on $[a, b]$. Now the cone K in X is normal, so the order interval $[a, b]$ is norm-bounded in X. Hence there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. As f is continuous on compact $J \times [-r, r] \times [-r, r]$, it attains its maximum, say M. Therefore for any subset S of $[a, b]$, we have:

$$\begin{aligned}
\|A(S)\|_{\mathcal{P}} &= \sup\{\|Ax\| : x \in S\} \\
&= \sup\left\{\sup_{t\in J} |f(t, x(t), x(\mu(t)))| : x \in S\right\} \\
&\leq \sup\left\{\sup_{t\in J} |f(t, x, y)| : x, y \in [-r, r]\right\} \\
&\leq M.
\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the function $f(t, x, y)$ is uniformly continuous on $[0, T] \times [-r, r] \times [-r, r]$. Therefore for any $t, \tau \in [0, T]$, we have:

$$|f(t, x, y) - f(\tau, x, y)| \to 0 \text{ as } t \to \tau$$

for all $x, y \in [-r, r]$. Similarly for any $x_1, x_2, y_1, y_2 \in [-r, r]$

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \to 0 \text{ as } (x_1 \to y_1), (x_2 \to y_2)$$

for all $t \in [0, T]$. Hence any $t, \tau \in [0, T]$ and for any $x \in S$ one has

$$\begin{aligned}
|Ax(t) - Ax(\tau)| &= |f(t, x(t), x(\mu(t))) - f(\tau, x(\tau), x(\mu(\tau)))| \\
&\leq |f(t, x(t), x(\mu(t))) - f(\tau, x(t), x(\mu(t))| \\
&\qquad + |f(\tau, x(t), x(\mu(t)) - f(\tau, x(\tau), x(\mu(\tau)))| \\
&\to 0 \quad \text{as} \quad t \to \tau
\end{aligned}$$

uniformly for all $x \in S$. This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[a, b]$.

Step II : Next we show that B is totally bounded operator on $[a, b]$. To finish, we shall show that $B(S)$ is uniformly bounded ad equi-continuous set in X for any subset S of $[a, b]$. Since the cone K in X is normal, the order interval $[a, b]$ is norm-bounded.

Let $y \in B(S)$ be arbitrary. Then,

$$y(t) = \int_0^T G_h(t,s) g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds$$

for some $x \in S$. By hypothesis (B_2), one has

$$\begin{aligned}|y(t)| &= \int_0^T G_h(t,s)\Big|g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\Big|\,ds \\ &\le M_h \int_0^T q(s)\,ds \\ &\le M_h \|q\|_{L^1}.\end{aligned}$$

Taking the supremum over t,

$$\|y\| \le M_h \|q\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Similarly let $t, \tau \in J$. To finish it is enough to show that y' is bounded on $[0,T]$. Now for any $t \in [0,T]$,

$$\begin{aligned}|y'(t)| &\le \Big|\int_0^T \frac{\partial}{\partial t} G_h(t,s)\Big|g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\Big|\,ds\Big| \\ &= \Big|\int_0^T |(-h(t))| G_h(t,s)\Big|g_h\Big(s, x(\theta(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\Big|\,ds\Big| \\ &\le H\, M_h \|q\|_{L^1} \\ &= c.\end{aligned}$$

where $H = \max_{t\in J} |h(t)|$. Hence for any $t, \tau \in [0,T]$ one has

$$|y(t) - y(\tau)| \le c\, |t - \tau| \to 0 \quad \text{as} \quad t \to \tau$$

uniformly for all $y \in B(S)$. This shows that $B(S)$ is a equi-continuous set of functions in $[a,b]$. for all $S \subset [a,b]$. Now $B(S)$ is a uniformly bounded and equi-continuous, so it is totally bounded by Arzelà-Ascoli theorem. Thus all the conditions of Theorem 3.12 are satisfied and hence an application of it yields that the PBVP (1.1) has a maximal and a minimal positive solution on J. □

Theorem 3.17. *Suppose that the assumptions (A_0)-(A_1), (A_3), (B_0)-(B_2), (B_4)-(B_5) and (C_1)-(C_2) hold. Furthermore, if*

$$L M_h \|q\|_{L^1} < 1,$$

where q is given in Remark 3.5 and $L = \max_{t\in J} \ell(t)$, then the PBVP (1.1) has a minimal and a maximal positive solution on J.

Proof. Now PBVP (1.1) is equivalent to integral equation (2.7) on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on X by (2.11) and (2.12) respectively. Then integral equation (2.7) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [a,b] \to K$. Note that the conditions

(B_1), (B_2) and (B_3) provides $a \leq Aa\,Ba$ and $Ab\,Bb \leq b$. Since the cone K in X is normal, $[a,b]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 3.11 and Theorem 3.16 that the operator A is a Lipschitz with the Lipschitz constant $\alpha = L$ and B is totally bounded with $M = \|B([a,b])\| = M_h\|q\|_{L^1}$ on $[a,b]$. Since $\alpha M = L\,M_h\|q\|_{L^1} < 1$, the desired conclusion follows by an application of Theorem 3.13. □

4. AN EXAMPLE

Given the closed and bounded interval $J = [0,\pi]$ in $\mathbb{R}$, consider the first order periodic boundary value problem of FBVP,

(4.1)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{1+\frac{\sin t}{12}\left(|x(t)|+|x(t^2/\pi)|\right)}\right] &= -\left(\frac{x(t)}{1+\frac{\sin t}{12}\left(|x(t)|+|x(t^2/\pi)|\right)}\right) \\ &\quad + \overline{g}\Big(t, x(t/2), \int_0^{\pi-t} k(t,s,x(s/3))\,ds\Big) \text{ a.e. } t \in J \\ x(0) = x(\pi) & \end{aligned}\right\}$$

where, the functions $k : J \times J \times \mathbb{R} \to \mathbb{R}$, $\overline{g} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\theta, \mu, \sigma, \eta : J \to J$ are given by

$$\overline{g}(t,x,y) = \frac{p(t)x}{1+|x|} + |y|,$$

and

$$k(t,s,x) = \frac{x}{4\pi(1+|x|)}$$

where $p \in L^1(J,\mathbb{R})$. Here,

$$\mu(t) = t^2/\pi,\ \ \theta(t) = t/2,\ \ \sigma(t) = \pi - t, \quad \text{and} \quad \eta(t) = t/3$$

for $t \in J$. Clearly the functions $k : J \times J \times \mathbb{R} \to \mathbb{R}$ and $\theta, \mu, \sigma, \eta : J \to J$ are continuous with $\mu(0) = 0$ and $\mu(\pi) = \pi$.

Here, the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is defined by

$$f(t,x,y) = 1 + \frac{\sin t}{12}\left(|x| + |y|\right).$$

Obviously, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$. It is easy to verify that f is continuous and satisfies the hypotheses (A_0)-(A_3) on $J \times \mathbb{R} \times \mathbb{R}$ with $\ell(t) = \frac{1}{6}$ for all $t \in J$. To see this, let $x, y \in \mathbb{R}$, then we have

$$\begin{aligned}|f(t,x_1,x_2) - f(t,y_1,y_2)| &= \left| \left[1 + \frac{\sin t}{12}\left(|x_1| + |x_2|\right)\right] - \left[1 + \frac{\sin t}{12}\left(|y_1| + |y_2|\right)\right] \right| \\ &\le \frac{1}{12}\left|\left(\,|x_1| - |y_1| + |x_2| - |y_2|\,\right)\right| \\ &\le \frac{1}{12}\left(|x_1 - y_1| + |x_2 - y_2|\right) \\ &\le \frac{1}{6}\max\{|x_1 - y_1|\,,\,|x_2 - y_2|\}.\end{aligned}$$

Again the function $\overline{g}(t,x,y)$ is measurable in t for all $x, y \in \mathbb{R}$ and continuous in x and y almost everywhere for $t \in J$, and so $\overline{g}$ defines a Carathéodory mapping $\overline{g} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Furthermore, $g_1(= \overline{g})$ is also Carathéodory on $J \times \mathbb{R} \times \mathbb{R}$, and

$$\begin{aligned}|g_1(t,x,y)| &= \left| \frac{p(t)\,x(t)}{1 + |x(t)|} + \int_0^{\pi - t} \frac{x(s/3)}{4\pi(1 + |x(s/2)|)}\, ds \right| \\ &\le \left| \frac{p(t)\,x(t)}{1 + |x(t)|} \right| + \left| \int_0^{\pi - t} \frac{x(s/3)}{4\pi(1 + |x(s/2)|)}\, ds \right| \\ &\le |p(t)| + \frac{1}{4}\end{aligned}$$

Hence, the function g_1 is $L^1_{\mathbb{R}}$-Carathéodory and satisfies all the hypotheses (A_5) and (A_6) on $J \times \mathbb{R} \times \mathbb{R}$ with $\gamma(t) = |p(t)| + \frac{1}{4}$ on J and $\psi(r) = 1$ for all $r \in \mathbb{R}^+$. Therefore, if $\|p\|_{L^1} < 5$ and $r = 2$, then by Theorem 3.6, then the FBVP (4.1) has a solution in $\overline{\mathcal{B}_2(0)}$ defined on J.

Remark 4.1. While concluding this paper, we mention that our existence results of this paper can be extended to the infinite dimensional Banach algebras with appropriate modifications. Also existence results of this paper, include the existence results for the functional nonlinear quadratic differential equations with periodic boundary conditions, viz.,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(\mu(t)))}\right] &= g(t,x(\eta(t)))\ \text{a.e. } t \in J, \\ x(0) &= x(T). \end{aligned}\right\} \tag{4.2}$$

which is again new to the literature. A special case of the PBVP (4.2) with $\mu(t) = t = \eta(t)$ has been discussed in Dhage *et al* [12]. In a nutshell, our problem as well as the established results are quite general in the theory of periodic boundary value problems of ordinary nonlinear differential equations.

REFERENCES

[1] S. R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York-London, 1974.

[2] B. C. Dhage, *On α-condensing mappings in Banach algebras*, The Math. Student **63**(1994), 146–152.

[3] B. C. Dhage, *Fixed point theorems in ordered Banach algebras and applications*, PanAmer. Math. J. **9**(4) (1999), 93–102.

[4] B. C. Dhage, *Some nonlinear alternatives in Banach algebras with applications II*, Kyungpook Math. J. **45** (2005), 281–292.

[5] B. C. Dhage, *On a fixed point theorem in Banach algebras with applications*, Appl. Math. Lett. **18** (2005), 273–280.

[6] B. C. Dhage, *Some algebraic fixed point theorems for multi-valued mappings with applications*, Disc. Math. Differential Inclusions, Control & Optimization **26** (2006), 5–55.

[7] B. C. Dhage, *Nonlinear functional boundary value problems in Banach algebras involving Carathéodories*, Kyungpook Math. J. **46** (2006), 527–541.

[8] B. C. Dhage and S. K. Ntouyas, *Existence theorems for nonlinear functional boundary value problems*, Math. Sci. Res. J. **6** (8) (2002), 380–390.

[9] B. C. Dhage, U. P. Dolhare and S. K. Ntouyas, *Existence theorems for nonlinear first order functional differential equations in Banach algebras*, Comm. Appl. Nonlinear Anal. **10** (4) (2003), 59–69.

[10] B. C. Dhage and D. O'Regan, *A fixed point theorem in Banach algebras with applications to nonlinear integral equation*, Functional Diff. Equations **7**(3-4)(2000), 259–267.

[11] B. C. Dhage, S. N. Salunkhe, Ravi Agarwal and W. Zhang, *A functional differential equation in Banach algebras*, Math. Inequ. Appl. **8** (1) (2005), 89–99.

[12] B. C. Dhage, J. Henderson and S. K. Ntouyas, *A periodic boundary value problem of first order ordinary differential equations in Banach algebras*, Jour. Nonlinear Funct. Anal. and Diff. Equations **1** (2007), 103–120.

[13] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Verlag 2003.

[14] A. Granas, R. B. Guenther and J. W. Lee, *Some general existence principles for Carathéodory theory of nonlinear differential equations*, J. Math. Pures et Appl. **70** (1991), 153–196.

[15] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Spaces*, Academic Press, New York, 1988.

[16] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinues Differential Equations*, Marcel Dekker Inc., New York, 1994.

[17] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, New York, 1985.

[18] J. J. Nieto, *Basic theory for nonresonance impulsive periodic problems of first order*, J. Math. Anal. Appl. **205** (1997), 423–433.

[19] J. J. Nieto, *Periodic boundary value problems for first order impulsive ODEs*, Nonlinear Anal. **51** (2002), 1223–1232.

[20] P. Omari and F. Zanolin, *Remarks on periodic solutions for first order nonlinear differential systems*, Boll. Un. Mat. Ital. B (6) **2** (1983), 207–218.

[21] Tadeusz Jankowski, *BVPs for functional differential equations*, Dynam. Systems Appl. 9 (2000), no. 3, 445–455.

[22] E. Zeidler, *Nonlinear Functional Analysis: Part I,* Springer Verlag, New York, 1985.