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ON SOLUTIONS OF A QUADRATIC HAMMERSTEIN INTEGRAL EQUATION ON AN UNBOUNDED INTERVAL

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ABSTRACT. In this paper we examine the solvability of a nonlinear quadratic Hammerstein integral equation. This equation is considered in the Banach space of real functions which are defined, bounded and continuous on the real half-line. Using the idea of measures of noncompactness with the classical Schauder fixed point theorem we show that the equation has solutions which vanish at infinity.

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1. INTRODUCTION

The object of this paper is to study the nonlinear quadratic Hammerstein integral equation

$$x(t) = p(t) + f(t, x(t)) \int_0^\infty g(t,\tau)h(\tau, x(\tau))d\tau \ , \quad t \geq 0 \ . \tag{1.1}$$

Notice that Eq. (1.1) is a generalization of the classical Hammerstein integral equation on bounded interval having the form

$$x(t) = p(t) + \int_a^b g(x,\tau)h(\tau, x(\tau))d\tau \tag{1.2}$$

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and its quadratic counterpart which has the form

$$x(t) = p(t) + f(t, x(t)) \int_a^b g(t,\tau)h(\tau, x(\tau))d\tau \ . \tag{1.3}$$

It is worthwhile mentioning that the integral equations (1.2) and (1.3) arise in several applications in real world problems. For example, some problems considered in vehicular traffic theory, biology, queuing theory, the theory of radiative transfer and kinetic theory of gases lead to the quadratic integral equations of the form (1.3) (see [7-10, 12]) while many problems considered in mechanics can be described with help of the classical Hammerstein integral equation of the form (1.2).

The quadratic integral equation (1.1) will be considered here in the Banach space of real functions which are defined, bounded and continuous on the real half-line $\mathbb{R}_+ = [0, \infty)$. Our analysis uses the idea of measures of noncompactness with the classical Schauder fixed point theorem. Such an approach enables us to prove an existence result concerning Eq. (1.1) under rather general conditions which are easy to verify in concrete situations. The existence result obtained in this paper generalizes several ones in the literature [1, 2, 9, 10, 13, 14, 16, 17] and in the last section of this paper we compare our results with those obtained in [6].

Also we note that the approach presented in this paper allows us to prove that Eq. (1.1) has solutions vanishing at infinity.

2. NOTATION AND AUXILIARY FACTS

In this section we present some facts concerning measures of noncompactness. Let $(E, ||\cdot||)$ be an infinite dimensional Banach space with the zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. For a subset X of E we write $\overline{X}$, Conv$X$ in order to denote the closure and convex closure of X, respectively. The family of all nonempty and bounded subsets of E is denoted by $\mathfrak{M}_E$ and its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_E$.

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1^o The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset E$.
2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3^o $\mu(\overline{X}) = \mu(X)$.
4^o $\mu(\text{Conv}X) = \mu(X)$.
5^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \dots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in 1^o is called *the kernel of the measure of noncompactness* μ. Let us observe that the intersection set X_∞ from 6^o belongs to $\ker \mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for every n then we have that $\mu(X_\infty) = 0$. This simple observation will be crucial later. Further facts concerning measures of noncompactness and its properties may be found in [5].

In what follows we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+ = [0, \infty)$. The space $BC(\mathbb{R}_+)$ is furnished with the standard norm

$$||x|| = \sup\{|x(t)| : \ t \geq 0\} \ .$$

Let us describe the measure of noncompactness in $BC(\mathbb{R}_+)$ which will be used in further investigations. This measure was introduced in [5] (cf. also [4]).

Let us fix a nonempty bounded subset X of $BC(\mathbb{R}_+)$ and a positive number T. For $x \in X$ and $\varepsilon \geq 0$ denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \varepsilon\} \ .$$

Further, let us put:

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : \ x \in T\} \ ,$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon) \ ,$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X) \ .$$

Moreover, let us consider the quantity:

$$\beta(X) = \lim_{T \to \infty} \{\sup_{x \in X} \{\sup\{|x(t)| : \ t \geq T\}\}\} \ .$$

Finally, define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \beta(X) \ . \tag{2.1}$$

We know [4, 5] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$. The kernel $\ker \mu$ of this measure contains nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and vanish at infinity uniformly with respect to the set X, i.e. for any $\varepsilon > 0$ there exists $T > 0$ such that $|x(t)| \leq \varepsilon$ for $t \geq T$ and $x \in X$. This property permits us to characterize solutions of Eq. (1.1) considered in the next section.

3. MAIN RESULT

In this section let us consider the existence and asymptotic behaviour of solutions of the quadratic Hammerstein integral equation (1.1).

We will investigate Eq. (1.1) under the following assumptions:

(i) $p : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $p(t) \to 0$ as $t \to \infty$.

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x) - f(t,y)| \le m(t)|x-y|$$

for all $x, y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(iii) $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function.

(iv) $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(t,x)| \le a(t)b(|x|)$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(v) The function $\tau \to a(\tau)|g(t,\tau)|$ is integrable over $\mathbb{R}_+$ for any fixed $t \in \mathbb{R}_+$.

(vi) The functions $G, F, M : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas

$$G(t) = \int_0^\infty a(\tau)|g(t,\tau)|d\tau \ ,$$

$$F(t) = |f(t,0)| \int_0^\infty a(\tau)|g(t,\tau)|d\tau \ ,$$

$$M(t) = m(t) \int_0^\infty a(\tau)|g(t,\tau)|d\tau$$

are bounded on $\mathbb{R}_+$ and the function $F(t)$ vanishes at infinity i.e. $\lim_{t\to\infty} F(t) = 0$.

(vii) The following equalities hold:

$$\lim_{T\to\infty} \left\{ \sup \left\{ |f(t,0)| \int_T^\infty a(\tau)|g(t,\tau)|d\tau : t \in \mathbb{R}_+ \right\} \right\} = 0 \ .$$

$$\lim_{T\to\infty} \left\{ \sup \left\{ m(t) \int_T^\infty a(\tau)|g(t,\tau)|d\tau : \ t \in \mathbb{R}_+ \right\} \right\} = 0 \ ,$$

Now, keeping in mind assumption (vi) we may define the following finite constants:

$$\overline{G} = \sup\{G(t) : \ t \in \mathbb{R}_+\} \ ,$$

$$\overline{F} = \sup\{F(t) : \ t \in \mathbb{R}_+\} \ ,$$

$$\overline{M} = \sup\{M(t) : \ t \in \mathbb{R}_+\} \ .$$

We will also assume the following hypothesis:

(viii) There exists a positive solution r_0 of the inequality

$$||p|| + \overline{M} r b(r) + \overline{F} b(r) \leq r$$

such that $\overline{M} b(r_0) < 1$.

Remark 3.1. Let us observe that the inequality $\overline{M} b(r_0) < 1$ from assumption (viii) is satisfied provided Eq. (1.1) is not trivial. Indeed, let r_0 be a positive solution of the first inequality from (viii), i.e.

$$||p|| + \overline{M} r_0 b(r_0) + \overline{F} b(r_0) \leq r_0 \ .$$

Hence we get

$$\overline{M} r_0 b(r_0) \leq r_0 - ||p|| - \overline{F} b(r_0)$$

and consequently

$$\overline{M} b(r_0) \leq 1 - \frac{||p||}{r_0} - \frac{\overline{F} b(r_0)}{r_0} \ .$$

From the last inequality follows our assertion.

Now, we can formulate our main result.

Theorem 3.2. *Under the assumptions (i)-(viii) Eq.* (1.1) *has at least one solution* $x = x(t)$ *in the space* $BC(\mathbb{R}_+)$. *Moreover, all solutions of Eq.* (1.1) *belonging to the ball* B_{r_0} *vanish uniformly at infinity, where* r_0 *is a number appearing in assumption (viii).*

Proof. Consider the operator H defined on the space $BC(\mathbb{R}_+)$ by the formula

$$(Hx)(t) = p(t) + f(t, x(t)) \int_0^\infty g(t,\tau) h(\tau, x(\tau)) d\tau, \ t \in \mathbb{R}_+ \ .$$

Notice that assumptions (i)-(v) imply that the function Hx is well defined and continuous on $\mathbb{R}_+$ for any function $x \in BC(\mathbb{R}_+)$.

Further, utilizing our assumptions, for an arbitrary fixed $t \in \mathbb{R}_+$ we obtain

$$\begin{aligned} |(Hx)(t)| &\leq |p(t)| + |f(t,x(t))| \int_0^\infty |g(t,\tau)||h(\tau, x(\tau))| d\tau \\ &\leq |p(t)| + [|f(t,x(t)) - f(t,0)| + |f(t,0)|] \int_0^\infty |g(t,\tau)| a(\tau) b(|x(\tau)|) d\tau \\ &\leq |p(t)| + [m(t)|x(t)| + |f(t,0)|] \int_0^\infty a(\tau)|g(t,\tau)| b(||x||) d\tau = |p(t)| \end{aligned}$$

$$+ b(||x||)|x(t)|m(t)\int_0^\infty a(\tau)|g(t,\tau)|d\tau + b(||x||)|f(t,0)|\int_0^\infty a(\tau)|g(t,\tau)|d\tau$$

$$\leq |p(t)| + b(||x||)|x(t)|M(t) + b(||x||)F(t)\ . \tag{3.1}$$

Hence we obtain

$$|(Hx)(t)| \leq ||p|| + \overline{M}||x||b(||x||) + \overline{F}b(||x||)\ .$$

The above estimate permits us to infer that the function Hx is bounded on the interval $\mathbb{R}_+$. Thus, we deduce that the operator H transforms the space $BC(\mathbb{R}_+)$ into itself. Moreover, from this estimate we obtain the following inequality

$$||Hx|| \leq ||p|| + \overline{M}||x||b(||x||) + \overline{F}b(||x||)\ .$$

This inequality in conjunction with assumption (viii) ensures the existence of a positive number r_0 such that $\overline{M}b(r_0) < 1$ and the operator H transforms the ball B_{r_0} into itself.

Further, let us take a nonempty subset X of the ball B_{r_0}. Next, fix arbitrarily $T > 0$ and $\varepsilon > 0$. Choose a function $x \in X$ and take $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. Then, keeping in mind the assumptions, we get:

$$\begin{aligned}
|(Hx)(t) - (Hx)(s)| &\leq |p(t) - p(s)| \\
&+ \left| f(t,x(t))\int_0^\infty g(t,\tau)h(\tau,x(\tau))d\tau - f(s,x(s))\int_0^\infty g(t,\tau)h(\tau,x(\tau))d\tau \right| \\
&+ \left| f(s,x(s))\int_0^\infty g(t,\tau)h(\tau,x(\tau))d\tau - f(s,x(s))\int_0^\infty g(s,\tau)h(\tau,x(\tau))d\tau \right| \\
&\leq \omega^T(p,\varepsilon) + |f(t,x(t)) - f(s,x(s))|\int_0^\infty |g(t,\tau)||h(\tau,x(\tau))|d\tau \\
&+ |f(s,x(s))|\int_0^\infty |g(t,\tau) - g(s,\tau)||h(\tau,x(\tau))|d\tau \leq \omega^T(p,\varepsilon) \\
&+ [|f(t,x(t)) - f(t,x(s))| + |f(t,x(s)) - f(s,x(s))|]\int_0^\infty |g(t,\tau)|a(\tau)b(||x||)d\tau \\
&+ [|f(s,x(s)) - f(s,0)| + |f(s,0)|]\int_0^\infty |g(t,\tau) - g(s,\tau)|a(\tau)b(||x||)d\tau \\
&\leq \omega^T(p,\varepsilon) + [m(t)|x(t) - x(s)| + \omega^T_{r_0}(f,\varepsilon)]b(r_0)\int_0^\infty a(\tau)|g(t,\tau)|d\tau
\end{aligned}$$

$$+ [m(s)|x(s)| + |f(s,0)|]b(r_0) \int_0^\infty |g(t,\tau) - g(s,\tau)|a(\tau)d\tau$$

$$\leq \omega^T(p,\varepsilon) + b(r_0)\omega^T(x,\varepsilon)m(t) \int_0^\infty a(\tau)|g(t,\tau)|d\tau$$

$$+ b(r_0)\omega^T_{r_0}(f,\varepsilon) \int_0^\infty a(\tau)|g(t,\tau)|d\tau$$

$$+ [m(s)r_0 + |f(s,0)|]b(r_0) \left[\int_0^T a(\tau)|g(t,\tau) - g(s,\tau)|d\tau \right.$$

$$\left. + \int_T^\infty a(\tau)|g(t,\tau) - g(s,\tau)|d\tau \right] \leq \omega^T(p,\varepsilon) + b(r_0)M(t)\omega^T(x,\varepsilon)$$

$$+ b(r_0)G(t)\omega^T_{r_0}(f,\varepsilon) + r_0 b(r_0)m(s) \int_0^T a(\tau)\omega_1^T(g,\varepsilon)d\tau$$

$$+ b(r_0)|f(s,0)| \int_0^T a(\tau)\omega_1^T(g,\varepsilon)d\tau$$

$$+ r_0 b(r_0)m(s) \int_T^\infty a(\tau)[|g(t,\tau)| + |g(s,\tau)|]d\tau$$

$$+ b(r_0)|f(s,0)| \int_T^\infty a(\tau)[|g(t,\tau)| + |g(s,\tau)|]d\tau$$

$$\leq \omega^T(p,\varepsilon) + \overline{M}b(r_0)\omega^T(x,\varepsilon) + \overline{G}b(r_0)\omega^T_{r_0}(f,\varepsilon)$$

$$+ (r_0 m_T + F_T)b(r_0)\omega_1^T(g,\varepsilon) \int_0^T a(\tau)d\tau$$

$$+ r_0 b(r_0)m(s) \int_T^\infty a(\tau)|g(s,\tau)|d\tau + r_0 b(r_0)m(s) \int_T^\infty a(\tau)|g(t,\tau)|d\tau$$

$$+ b(r_0)|f(s,0)| \int_T^\infty a(\tau)|g(s,\tau)|d\tau + b(r_0)|f(s,0)| \int_T^\infty a(\tau)|g(t,\tau)|d\tau \ , \tag{3.2}$$

where we denoted

$$\omega^T_{r_0}(f,r_0) = \sup\{|f(t,x) - f(s,x)| : \ t,s \in [0,T], \ |t-s| \leq \varepsilon, \ x \in [-r_0,r_0]\} \ ,$$
$$\omega_1^T(g,\varepsilon) = \sup\{|g(t,\tau) - g(s,\tau)| : \ t,s,\tau \in [0,T], \ |t-s| \leq \varepsilon\} \ ,$$

$$m_T = \sup\{m(t):\ t \in [0,T]\}\ ,$$
$$F_T = \sup\{|f(t,0)|:\ t \in [0,T]\}\ .$$

Now, we have the following estimate:

$$\begin{aligned} m(s)\int_T^\infty a(\tau)|g(t,\tau)|d\tau &\le [|m(s)-m(t)|+|m(t)|]\int_T^\infty a(\tau)|g(t,\tau)|d\tau \\ &\le \omega^T(m,\varepsilon)\int_T^\infty a(\tau)|g(t,\tau)|d\tau + m(t)\int_T^\infty a(\tau)|g(t,\tau)|d\tau \\ &\le \omega^T(m,\varepsilon)\int_0^\infty a(\tau)|g(t,\tau)|d\tau + m(t)\int_T^\infty a(\tau)|g(t,\tau)|d\tau \\ &\le \overline{G}\omega^T(m,\varepsilon) + m(t)\int_T^\infty a(\tau)|g(t,\tau)|d\tau\ . \end{aligned} \tag{3.3}$$

Similarly, we get:

$$\begin{aligned} |f(s,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau &\le [|f(s,0)-f(t,0)|+|f(t,0)|]\int_T^\infty a(\tau)|g(t,\tau)|d\tau \\ \overline{\omega}^T(f,\varepsilon)\int_T^\infty a(\tau)|g(t,\tau)|d\tau &+ |f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau \\ &\le \overline{G}\overline{\omega}^T(f,\varepsilon) + |f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau\ , \end{aligned} \tag{3.4}$$

where we denoted

$$\overline{\omega}^T(f,\varepsilon) = \sup\{|f(t,0)-f(s,0)|:\ t,s \in [0,T],\ |t-s| \le \varepsilon\}\ .$$

In what follows let us observe that linking (3.2), (3.3), (3.4) and taking into account the uniform continuity of the functions $p(t),\ m(t)$ on the interval $[0,T]$ and the uniform continuity of the functions $f(t,x),\ g(t,\tau)$ on the sets $[0,T]\times[-r_0,r_0]$, $[0,T]\times[0,T]$, respectively, we obtain the following inequality

$$\begin{aligned} \omega_0^T(HX) \le\ & \overline{M}b(r_0)\omega_0^T(X) \\ & + r_0 b(r_0)\left\{m(s)\int_T^\infty a(\tau)|g(s,\tau)|d\tau + m(t)\int_T^\infty a(\tau)|g(t,\tau)|d\tau\right\} \\ & + b(r_0)\left\{|f(s,0)|\int_T^\infty a(\tau)|g(s,\tau)|d\tau + |f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau\right\}\ . \end{aligned}$$

The above inequality with assumption (vii) implies

$$\omega_0(HX) \le \overline{M}b(r_0)\omega_0(X)\ . \tag{3.5}$$

Now, take an arbitrary function $x \in X$ and a number $T > 0$. Then, from estimate (3.1) we obtain

$$\begin{aligned}\sup\{|(Hx)(t)| : t \ge T\} &\le \sup\{|p(t)| : t \ge T\} \\ &+ b(||x||)\overline{M}\sup\{|x(t)| : t \ge T\} + b(||x||)\sup\{F(t) : t \ge T\}\ .\end{aligned}$$

Hence, in view of assumptions (i) and (vi) we get

$$\beta(HX) \le \overline{M}b(r_0)\beta(X)\ , \tag{3.6}$$

where the function β was defined in Section 2.

Further, linking (3.5), (3.6) and keeping in mind the definition of the measure of noncompactness given by formula (2.1), we obtain

$$\mu(HX) \le \overline{M}b(r_0)\mu(X)\ . \tag{3.7}$$

In the sequel let us consider the sequence of sets $(B^n_{r_0})$, where $B^1_{r_0} = \mathrm{Conv}H(B_{r_0})$, $B^2_{r_0} = \mathrm{Conv}H(B^1_{r_0})$ and so on. Observe that this sequence is decreasing i.e. $B^{n+1}_{r_0} \subset B^n_{r_0}$ for $n = 1, 2, \ldots$. Moreover, $B^1_{r_0} \subset B_{r_0}$. In addition we have that the sets of this sequence are closed, convex and nonempty. On the other hand, in view of (3.7) we get

$$\mu(B^n_{r_0}) \le k^n\mu(B_{r_0})\ ,$$

for any $n = 1, 2, \ldots$, where $k = \overline{M}b(r_0)$. Taking into account that $k < 1$ (cf. assumption (viii)), from the above estimate we infer that

$$\lim_{n\to\infty} \mu(B^n_{r_0}) = 0\ .$$

Hence, taking into account Definition 2.1 we deduce that the set $Y = \bigcap_{n=1}^{\infty} B^n_{r_0}$ is nonempty, bounded, closed and convex. Moreover, the set Y is a member of the kernel $\ker\mu$ of the measure of noncompactness μ (cf. remark made after Definition 2.1). Let us also observe that the operator H transforms the set Y into itself.

In what follows we show that H is continuous on the set Y. To do this let us fix a number $\varepsilon > 0$ and take arbitrary functions $x, y \in Y$ such that $||x - y|| \le \varepsilon$. Keeping in mind the fact that $Y \in \ker\mu$ and the structure of sets belonging to $\ker\mu$ (cf. Section 2) we can find a number $T > 0$ such that for each $z \in Y$ and $t \ge T$ we have that $|z(t)| \le \varepsilon$. Since $H : Y \to Y$ we have that $Hx,\ Hy \in Y$. Thus, for $t \ge T$ we get

$$|(Hx)(t) - (Hy)(t)| \le |(Hx)(t)| + |(Hy)(t)| \le 2\varepsilon\ . \tag{3.8}$$

On the other hand, taking an arbitrary number $t \in [0,T]$ and applying the assumptions, we obtain:

$$
\begin{aligned}
|(Hx)(t)-(Hy)(t)| &\le |f(t,x(t))-f(t,y(t))| \int_0^\infty |g(t,\tau)||h(\tau,x(\tau))|d\tau \\
&\quad + |f(t,y(t))| \int_0^\infty |g(t,\tau)||h(\tau,x(\tau))-h(\tau,y(\tau))|d\tau \\
&\le \varepsilon m(t) \int_0^\infty |g(t,\tau)|a(\tau)b(r_0)d\tau \\
&\quad + (m(t)|y(t)|+|f(t,0)|) \int_0^\infty |g(t,\tau)||h(\tau,x(\tau))-h(\tau,y(\tau))|d\tau \\
&\le \varepsilon \overline{M} b(r_0) + (r_0 m(t)+|f(t,0)|) \left\{ \int_0^T |g(t,\tau)||h(\tau,x(\tau))-h(\tau,y(\tau))|d\tau \right. \\
&\quad \left. + \int_T^\infty |g(t,\tau)|[|h(\tau,x(\tau))|+|h(\tau,y(\tau))|]d\tau \right\} \\
&\le \varepsilon \overline{M} b(r_0) + (r_0 m(t)+|f(t,0)|) \left\{ \int_0^T |g(t,\tau)|\omega_{r_0}^T(h,\varepsilon)d\tau \right. \\
&\quad \left. + \int_T^\infty a(\tau)|g(t,\tau)|2b(r_0)d\tau \right\} \le \varepsilon \overline{M} b(r_0) \\
&\quad + (r_0 M_T + F_T)T g_T \omega_{r_0}^T(h,\varepsilon) + 2r_0 b(r_0) m(t) \int_T^\infty a(\tau)|g(t,\tau)|d\tau \\
&\quad + 2b(r_0)|f(t,0)| \int_T^\infty a(\tau)|g(t,\tau)|d\tau \ , \qquad (3.9)
\end{aligned}
$$

where we denoted

$$g_T = \max\{|g(t,\tau)| : t,\tau \in [0,T]\} \ ,$$

$$\omega_{r_0}^T(h,\varepsilon) = \sup\{|h(t,x)-h(t,y)| : \ t \in [0,T], \ x,y \in [-r_0,r_0], \ |x-y| \le \varepsilon\} \ .$$

Observe that $\omega_{r_0}^T(h,\varepsilon) \to 0$ as $\varepsilon \to 0$ which is a simple consequence of the uniform continuity of the function $h(t,x)$ on the set $[0,T]\times[-r_0,r_0]$. Moreover, in view of assumption (vii) we can choose T in such a way that two last terms of the estimate (3.9) are sufficiently small.

Now, let us notice that linking (3.8), (3.9) and the above established facts we conclude that the operator H is continuous on the set Y.

Finally taking into account all the above properties of the set Y and the operator $H : Y \to Y$ and using the classical Schauder fixed point theorem we infer that the operator H has at least one fixed point x in the set Y. Obviously the function $x = x(t)$ is a solution of Eq. (1.1). Moreover, in view of the fact that $Y \in \ker \mu$ we have that $x(t) \to 0$ as $t \to \infty$.

On the other hand let us notice that if x is an arbitrary fixed point x of the operator H such that $x \in B_{r_0}$, then we can easily see that $x \in Y$. This proves the last assertion of our theorem.

Thus the proof is complete. □

4. REMARKS, EXAMPLES AND COMPARISON WITH OTHER RESULTS

In this section we compare our result with the result proved recently in [6]. First of all let us notice that in [6] the following restrictive hypothesis was assumed:

(iv′) $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous on every rectangle of the form $\mathbb{R}_+ \times [-q, q]$.

Obviously in our assumption (iv) we require only the continuity of h on $\mathbb{R}_+ \times \mathbb{R}$.

Let us also observe that another restrictive assumption imposed in [6] is the following hypothesis concerning the function $g = g(t, \tau)$:

(iii′) $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function and there exist continuous functions $k, l : \mathbb{R}_+ \to \mathbb{R}_+$ such that the functions $l(t)$ and $a(t)l(t)$ are integrable over $\mathbb{R}_+$ and the following inequality

$$|g(t, s)| \leq k(t)l(s)$$

is satisfied for $t, s \in \mathbb{R}_+$. Moreover, we assume that $k(t) \to 0$ as $t \to \infty$ and the function $m(t)k(t)$ is bounded on the interval $\mathbb{R}_+$.

Notice that our assumptions (iii), (v), (vi) and (vii) being the counterparts of assumption (iii′) are more general. Indeed, we do not require the boundedness of the function $g = g(t, \tau)$ by the product of two functions with separable variables. On the other hand it can be shown that assumption (iii′) implies that all the assumptions (iii), (v), (vi) and (vii) are satisfied. We omit the easy details.

In order to illustrate the result contained in Theorem 3.2 let us consider the following quadratic Hammerstein integral equation

$$x(t) = te^{-4t} + \left(tx(t) + \frac{t}{t^2 + 16}\right) \int_0^\infty \frac{t^2 e^{-\tau}}{t^2 + 1} \sqrt{|x(\tau)|} d\tau \ . \tag{4.1}$$

Observe that the above equation was considered as an example in [6] (cf. Eq. (12) in the mentioned paper). Obviously, Eq. (4.1) is a special case of Eq. (1.1) if we put (similarly as in [6]) $p(t) = te^{-4t}$, $h(t,x) = t\sqrt{|x|}$ and

$$f(t,x) = tx + \frac{t}{t^2+16} ,$$

$$g(t,\tau) = \frac{te^{-\tau}}{t^2+1} .$$

In [6] the authors overlooked the fact that the function $h(t,x)$ does not satisfy assumption (iv′) mentioned above. So the result contained in [6] cannot be applied to Eq. (4.1).

However we show using Theorem 3.2 that Eq. (4.1) has solutions in the space $BC(\mathbb{R}_+)$ vanishing at infinity.

Indeed, we have that $m(t) = t$, $a(t) = t$, $b(r) = \sqrt{r}$ and $f(t,0) = t/(t^2+16)$. Obviously assumptions (i)-(iv) of Theorem 3.2 are satisfied.
In order to show that assumption (v) is satisfied let us notice that

$$\int_0^\infty a(\tau)|g(t,\tau)|d\tau = \int_0^\infty \tau\frac{te^{-\tau}}{t^2+1}d\tau = \frac{t}{t^2+1}\int_0^\infty \tau e^{-\tau}d\tau = \frac{t}{t^2+1} .$$

This shows that assumption (v) holds and $G(t) = t/(t^2+1)$. Consequently, $\overline{G} = 1/2$.

Further, we get:

$$F(t) = |f(t,0)|\int_0^\infty a(\tau)|g(t,\tau)|d\tau = \frac{t}{t^2+16}\cdot\frac{t}{t^2+1} .$$

Hence we obtain that $F(t) \to 0$ as $t \to \infty$. Moreover, using standard methods of differential calculus we derive that $\overline{F} = 1/25$. Also we have:

$$M(t) = m(t)\int_0^\infty a(\tau)|g(t,\tau)|d\tau = t\frac{t}{t^2+1} = \frac{t^2}{t^2+1} .$$

Hence we see that the function $M(t)$ is bounded and $\overline{M} = 1$. Further, let us fix arbitrarily $T > 0$. Then we obtain:

$$m(t)\int_T^\infty a(\tau)|g(t,\tau)|d\tau = t\int_T^\infty \tau\frac{te^{-\tau}}{t^2+1}d\tau = \frac{t^2}{t^2+1}\int_T^\infty \tau e^{-\tau}d\tau$$
$$= \frac{t^2}{t^2+1}(Te^{-T}+e^{-T}) \le Te^{-T}+e^{-T} .$$

Similarly, we get:

$$|f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau = \frac{t^2}{t^2+16}(Te^{-T}+e^{-T}) \le Te^{-T}+e^{-T} .$$

From the above estimates we infer that assumption (vii) holds.

Finally, let us consider the inequality from assumption (viii) which has the form

$$\frac{1}{4e} + r\sqrt{r} + \frac{1}{25}\sqrt{r} \leq r \ . \tag{4.2}$$

It is easy to check that the number $r_0 = 1/5$ satisfies the above inequality. Moreover, we have that

$$\overline{M}b(r_0) = \sqrt{1/5} < 1 \ .$$

This shows that all the assumptions of Theorem 3.2 hold. Hence we deduce that Eq. (4.1) has solutions in the space $BC(\mathbb{R}_+)$, belonging to the ball $B_{1/5}$ and vanishing at infinity. Moreover, all solutions of Eq. (4.1) belonging to the ball $B_{1/5}$ vanish at infinity.

It is worthwhile noticing that also the number $r_1 = 0.6$ satisfies the inequality (4.2) and $\overline{M}b(r_1) < 1$. Hence we conclude that all possible solutions of Eq. (4.1) belonging to the ball $B_{0.6}$ vanish at infinity.

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