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A NOTE ON GRÜSS TYPE INEQUALITIES ON TIME SCALES

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ABSTRACT. The purpose of this paper is to investigate some Grüss type inequalities on time scales. Our results unify some continuous inequalities and their corresponding discrete analogues. We also apply our results to the quantum calculus case.

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1. INTRODUCTION

In [3], Bohner and Matthews proved the time scales version of the Grüss inequality as follows: Let $a, b, s \in \mathbb{T}$, $f, g \in C_{rd}$ and $f, g : [a, b] \to \mathbb{R}$. Then for

$$m_1 \leq f(s) \leq M_1, \qquad m_2 \leq g(s) \leq M_2$$

we have

(1.1)

$$\left| \frac{1}{b-a} \int_a^b f^\sigma(s) g^\sigma(s) \Delta s - \frac{1}{(b-a)^2} \int_a^b f^\sigma(s) \Delta s \int_a^b g^\sigma(s) \Delta s \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

In this paper, we investigate some Grüss type inequalities on time scales, which unify some inequalities established by Pachpatte in [6]. We also apply our result to the special cases of continuous, discrete, and quantum calculus.

Further information on the time scales calculus can be found in [1, 2].

2. THE GRÜSS TYPE INEQUALITIES ON TIME SCALES

The main results are given in the following theorem.

Theorem 2.1. *Let $a, b \in \mathbb{T}$, $a < b$, $f, g \in C_{rd}$ and $f, g : [a, b] \to \mathbb{R}$ be $\Delta-$differentiable functions. Then*

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$$\begin{aligned}&\left|\tfrac{1}{b-a}\int_a^b f^\sigma(x)g^\sigma(x)\Delta x-\tfrac{1}{2(b-a)}[F\int_a^b g^\sigma(x)\Delta x+G\int_a^b f^\sigma(x)\Delta x]\right|\\ &\leq \tfrac{1}{4(b-a)}\left[\left(\int_a^b\left|g^\sigma(x)\right|\Delta x\right)\left(\int_a^b\left|f^\Delta(\sigma(x))\right|\Delta x\right)\right.\\ &\left.+\left(\int_a^b\left|f^\sigma(x)\right|\Delta x\right)\left(\int_a^b\left|g^\Delta(\sigma(x))\right|\Delta x\right)\right],\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\left|\tfrac{1}{b-a}\int_a^b f^\sigma(x)g^\sigma(x)\Delta x-\tfrac{1}{b-a}[F\int_a^b g^\sigma(x)\Delta x+G\int_a^b f^\sigma(x)\Delta x]+FG\right|\\ &\leq \tfrac{1}{4}\left(\int_a^b\left|f^\Delta(\sigma(x))\right|\Delta x\right)\left(\int_a^b\left|g^\Delta(\sigma(x))\right|\Delta x\right),\end{aligned}\tag{2.2}$$

for all $x\in\mathbb{T}$, *where*

$$F=\frac{f^\sigma(a)+f^\sigma(b)}{2}\qquad G=\frac{g^\sigma(a)+g^\sigma(b)}{2}.$$

Proof. We have the following identities for all $x\in\mathbb{T}$

$$f^\sigma(x)-F=\frac{1}{2}\left[\int_a^x f^\Delta(\sigma(t))\Delta t-\int_x^b f^\Delta(\sigma(t))\Delta t\right]\tag{2.3}$$

$$g^\sigma(x)-G=\frac{1}{2}\left[\int_a^x g^\Delta(\sigma(t))\Delta t-\int_x^b g^\Delta(\sigma(t))\Delta t\right].\tag{2.4}$$

Multiplying both sides of (2.3) and (2.4) by $g^\sigma(x)$ and $f^\sigma(x)$ respectively, adding the resulting identities and rewriting we obtain

$$\begin{aligned}&f^\sigma(x)g^\sigma(x)-\tfrac{1}{2}[g^\sigma(x)F+f^\sigma(x)G]\\ &=\tfrac{1}{4}\left[g^\sigma(x)\left[\int_a^x f^\Delta(\sigma(t))\Delta t-\int_x^b f^\Delta(\sigma(t))\Delta t\right]\right.\\ &\left.+f^\sigma(x)\left[\int_a^x g^\Delta(\sigma(t))\Delta t-\int_x^b g^\Delta(\sigma(t))\Delta t\right]\right].\end{aligned}\tag{2.5}$$

From (2.5) and using the properties of modulus we have

$$\begin{aligned}&\left|f^\sigma(x)g^\sigma(x)-\frac{1}{2}[g^\sigma(x)F+f^\sigma(x)G]\right|\\ &\leq\frac{1}{4}\left[\left|g^\sigma(x)\right|\int_a^b\left|f^\Delta(\sigma(t))\right|\Delta t+\left|f^\sigma(x)\right|\int_a^b\left|g^\Delta(\sigma(t))\right|\Delta t\right].\end{aligned}\tag{2.6}$$

Dividing both sides of (2.6) by $(b-a)$, then integrating both sides with respect to x over $[a,b]$, we get the required inequality in (2.1).

Multiplying the left sides and right sides of (2.3) and (2.4), we have

(2.7)

$$f^{\sigma}(x)g^{\sigma}(x)-[g^{\sigma}(x)F+f^{\sigma}(x)G]+FG$$

$$=\tfrac{1}{4}\left[\int\limits_a^x f^{\Delta}(\sigma(t))\Delta t-\int\limits_x^b f^{\Delta}(\sigma(t))\Delta t\right]\left[\int\limits_a^x g^{\Delta}(\sigma(t))\Delta t-\int\limits_x^b g^{\Delta}(\sigma(t))\Delta t\right].$$

From (2.7) and using the properties of modulus, we get

$$|f^{\sigma}(x)g^{\sigma}(x)-[g^{\sigma}(x)F+f^{\sigma}(x)G]+FG|$$

$$\leq\frac{1}{4}\left[|g^{\sigma}(x)|\int\limits_a^b\left|f^{\Delta}(\sigma(t))\right|\Delta t+|f^{\sigma}(x)|\int\limits_a^b\left|g^{\Delta}(\sigma(t))\right|\Delta t\right]. \tag{2.8}$$

Dividing both sides of (2.8) by $(b-a)$, then integrating both sides with respect to x over $[a,b]$, we prove the inequality in (2.2). □

If we apply the Grüss type inequalities to different time scales, we will get some well-known and some new results: (i) If we take $\mathbb{T}=\mathbb{R}$, the inequalities (2.1) and (2.2) are similar to those of the well-known inequalities due to Grüss and Cebysev, see [4, 5]. (ii) Let$\mathbb{T}=\mathbb{Z}$. Then discrete case form of the inequalities (2.1) and (2.2) are proved similarly in [6].

Corollary 1 (Quantum Calculus Case)**.** *Let* $\mathbb{T}=q^{\mathbb{Z}_0}=\{q^k : k\in\mathbb{N}_0\},\ q>1,\ a=q^m$ *and* $b=q^n$. *Then we get*

$$\left|\frac{\sum\limits_{k=m}^{n-1}q^k f(q^{k+1})g(q^{k+1})}{\sum\limits_{k=m}^{n-1}q^k}-\frac{1}{2\sum\limits_{k=m}^{n-1}q^k}\sum\limits_{k=m}^{n-1}q^k[Fg(q^{k+1})+Gf(q^{k+1})]\right|$$

$$\leq\frac{1}{4\sum\limits_{k=m}^{n-1}q^k}\left[\left(\sum_{k=m}^{n-1}q^k g(q^{k+1})\right)\left(\sum_{k=m}^{n-1}q^k\Delta f(q^{k+1})\right)\right.$$

$$\left.+\left(\sum_{k=m}^{n-1}q^k f(q^{k+1})\right)\left(\sum_{k=m}^{n-1}q^k\Delta g(q^{k+1})\right)\right],$$

and

$$\left|\frac{\sum\limits_{k=m}^{n-1}q^k f(q^{k+1})g(q^{k+1})}{\sum\limits_{k=m}^{n-1}q^k}-\frac{1}{2\sum\limits_{k=m}^{n-1}q^k}\sum\limits_{k=m}^{n-1}q^k[Fg(q^{k+1})+Gf(q^{k+1})]+FG\right|$$

$$\leq\tfrac{1}{4}\left(\sum_{k=m}^{n-1}q^k\Delta f(q^{k+1})\right)\left(\sum_{k=m}^{n-1}q^k\Delta g(q^{k+1})\right),$$

where

$$F=\frac{f(q^{m+1})+f(q^{n+1})}{2}\qquad G=\frac{g(q^{m+1})+g(q^{n+1})}{2}.$$

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