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THREE POSITIVE SOLUTIONS FOR A GENERALIZED STURM-LIOUVILLE MULTIPOINT BVP WITH DEPENDENCE ON THE FIRST ORDER DERIVATIVE

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ABSTRACT. In this paper, we are concerned with the following generalized Sturm-Liouville multipoint boundary value problem

$$u''(t) + h(t) f(t, u(t), u'(t)) = 0, \ 0 < t < 1,$$

$$au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

where $0 < \xi_1 < \cdots < \xi_{m-2} < 1 \ (m \geq 3), a, b, c, d \in [0, \infty), \ a_i, b_i \in (0, \infty) \ (i = 1, 2, \ldots, m-2)$ are constants satisfying some suitable conditions. Existence criteria for at least three positive solutions are established by using the fixed point theorem of Avery and Peterson. The interesting point is the nonlinear term f which is involved with the first order derivative explicitly.

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1. INTRODUCTION

In this paper we are interested in the existence of three positive solutions for the following generalized Sturm-Liouville multipoint boundary value problem (BVP)

$$u''(t) + h(t) f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $a, b, c, d \in [0, \infty), \ 0 < \xi_1 < \cdots < \xi_{m-2} < 1 \, (m \geq 3), \ a_i, \ b_i \in (0, \infty)$ are constants for $i = 1, 2, \ldots, m-2$.

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by I1'in and Moiseev [7]. Since then, there has been much attention paid on the study of nonlinear multipoint boundary value problems, see [1, 3, 4, 6, 8, 9, 10] and the references therein.

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There are many papers dealing with the existence of positive solutions for multi-point BVP, in which the nonlinear term f is independent of the first order derivative with different boundary conditions. In particular, Ma [10] established some existence results of positive solutions for the problem

$$(\mathcal{L}u)(t) + h(t) f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

where $(\mathcal{L}u)(t) = (p(t)u'(t))' - q(t)u(t)$, the main tool is the well-known Guo-Krasnoselskii's fixed point theorems [5].

In [4], by a new fixed point theorem, Guo and Ge gave sufficient conditions for the existence of at least one solution to the following three point boundary value problem

$$u'' + f(t, u, u') = 0, \ 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta).$$

Motivated by the works above, our purpose of this paper is to establish some sufficient conditions for the existence of three positive solutions to the problem (1.1) and (1.2).

The rest of the paper is organized as follows. In section 2, we provide some lemmas which are useful later. An important lemma and criteria for the existence of three positive solutions for the generalized Sturm-Liouville multipoint BVP (1.1) and (1.2) are established in section 3. Finally, in section 4, we give an example to illustrate our results.

For convenience, we list the following hypotheses:

(A_1) $\rho = ac + ad + bc > 0$, a_i, b_i satisfy $a > \sum_{i=1}^{m-2} a_i$, $c > \sum_{i=1}^{m-2} b_i$.

(A_2) $h \in C([0,1], [0,\infty))$ and there exists $t_0 \in [\sigma, 1-\sigma]$ such that $h(t_0) > 0$, where $\sigma \in (0, 1/2)$ is a constant, $f \in C([0,1] \times [0,\infty) \times (-\infty,\infty), [0,\infty))$.

2. PRELIMINARIES

In this section, we present some preliminaries and basic lemmas which are useful later.

Firstly, for convenience, we define

$$x(t) = at + b \text{ and } y(t) = d + c(1-t) \text{ for } t \in [0,1]$$

and denote

$$\Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} a_i x(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i y(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i x(\xi_i) & -\sum_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix}.$$

Then it is easy to see that $x(t)$ and $y(t)$ are the solution of the problems $x''(t) = 0, x(0) = b, x'(0) = a$ and $y''(t) = 0, y(1) = d, y'(1) = -c$ respectively.

Lemma 2.1. [10] *Assume* (A_1) *holds. If* $\Delta \neq 0$, *then for* $g \in C[0,1]$, *the problem*

$$u''(t) + g(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) g(s)\, ds + x(t) A(g) + y(t) B(g), \tag{2.3}$$

where

$$G(t,s) = \frac{1}{\rho} \begin{cases} (d + c(1-t))(as+b), & 0 \le s \le t \le 1, \\ (at+b)(d+c(1-s)), & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$A(g) := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s) g(s) ds & \rho - \sum_{i=1}^{m-2} a_i y(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s) g(s) ds & -\sum_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix}, \tag{2.5}$$

$$B(g) := \frac{1}{\Delta} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i x(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s) g(s) ds \\ \rho - \sum_{i=1}^{m-2} b_i x(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s) g(s) ds \end{vmatrix}. \tag{2.6}$$

For the sake of convenience, we give the following hypothesis.

(A_3) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2} a_i y(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i x(\xi_i) > 0$.

Lemma 2.2. *If* (A_1) *and* (A_3) *hold, then for* $g \in C[0,1]$ *with* $g \ge 0$, *the unique solution* u *of the problem (2.1) and (2.2) satisfies*

$$u(t) \ge 0 \text{ for } t \in [0,1] \text{ and } \min_{\sigma \le t \le 1-\sigma} u(t) \ge \tau_1 \|u\|_0, \tag{2.7}$$

where $\tau_1 = min\left\{\frac{y(1-\sigma)}{y(0)}, \frac{x(\sigma)}{x(1)}\right\}$ *and* $\|u\|_0 := \max_{0 \le t \le 1} |u(t)|$.

Proof. From Lemma 2.1, we know that $G(t,s) \ge 0$. From (A_3), (2.5) and (2.6), $A(g) \ge 0$ and $B(g) \ge 0$. Thus by (2.3) we get that $u(t) \ge 0$ for $t \in [0,1]$.

In view of (2.4), it is easy to see that $G(t,s) = G(s,t)$, further

$$G(t,s) \le G(s,s), \quad t, s \in [0,1]. \tag{2.8}$$

For $t \in [\sigma, 1-\sigma]$, $s \in [0,1]$, we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{y(t)}{y(s)}, & s \le t, \\ \frac{x(t)}{x(s)}, & t \le s, \end{cases} \ge \begin{cases} \frac{y(1-\sigma)}{y(0)}, & s \le t, \\ \frac{x(\sigma)}{x(1)}, & t \le s, \end{cases} \ge \tau_1.$$

That is

$$G(t,s) \ge \tau_1 G(s,s). \tag{2.9}$$

By Lemma 2.1 and (2.8), we have

$$\begin{aligned}\|u\|_0 &= \max_{0\le t\le 1} u(t) = \max_{0\le t\le 1}\left(\int_0^1 G(t,s)g(s)ds + x(t)A(g) + y(t)B(g)\right)\\ &\le \int_0^1 G(s,s)\,g(s)\,ds + x(1)\,A(g) + y(0)\,B(g). \qquad (2.10)\end{aligned}$$

Hence, for $t\in[\sigma,1-\sigma]$, combining (2.9) and (2.10) with the monotonicity of x and y, we can conclude that

$$\begin{aligned}u(t) &= \int_0^1 G(t,s)\,g(s)\,ds + x(t)\,A(g) + y(t)\,B(g)\\ &\ge \int_0^1 \tau_1 G(s,s)\,g(s)\,ds + x(\sigma)\,A(g) + y(1-\sigma)\,B(g)\\ &\ge \tau_1\left[\int_0^1 G(s,s)\,g(s)\,ds + x(1)\,A(g) + y(0)\,B(g)\right] \ge \tau_1\|u\|_0 .\end{aligned}$$

The proof is complete. □

Let γ and θ be nonnegative continuous convex functionals on a cone K, α be a nonnegative continuous concave functional on K, β be a nonnegative continuous functional on K, and m_1,m_2,m_3,m_4 be positive numbers, we define the following convex sets

$$P(\gamma,m_4)=\{u\in K:\gamma(u)<m_4\};$$
$$P(\gamma,\alpha,m_2,m_4)=\{u\in K:m_2\le\alpha(u),\gamma(u)\le m_4\};$$
$$P(\gamma,\theta,\alpha,m_2,m_3,m_4)=\{u\in K:m_2\le\alpha(u),\theta(u)\le m_3,\ \gamma(u)\le m_4\};$$

and a closed set

$$Q(\gamma,\beta,m_1,m_4)=\{u\in K:m_1\le\beta(u),\gamma(u)\le m_4\}.$$

To prove our main results, we need the following fixed point theorem due to Avery and Peterson.

Lemma 2.3. [2] *Let K be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and β be a nonnegative continuous functional on K satisfying $\beta(\lambda u)\le\lambda\beta(u)$ for $0\le\lambda\le 1$, such that for some positive numbers ϵ and m_4,*

$$\alpha(u)\le\beta(u)\ \text{ and }\ \|u\|\le\epsilon\gamma(u),\ \text{ for all } u\in\overline{P(\gamma,m_4)}.$$

Suppose $T:\overline{P(\gamma,m_4)}\to\overline{P(\gamma,m_4)}$ is completely continuous and there are positive numbers m_1,m_2 and m_3 with $m_1<m_2$ such that

- (B_1) $\{u\in P(\gamma,\theta,\alpha,m_2,m_3,m_4):\alpha(u)>m_2\}\ne\emptyset$, *$\alpha(Tu)>m_2$ for $u\in P(\gamma,\theta,\alpha,m_2,m_3,m_4)$;*
- (B_2) *$\alpha(Tu)>m_2$ for $u\in P(\gamma,\alpha,m_2,m_4)$ with $\theta(Tu)>m_3$;*
- (B_3) *$0\notin Q(\gamma,\beta,m_1,m_4)$ and $\beta(Tu)<m_1$ for $u\in Q(\gamma,\beta,m_1,m_4)$ with $\beta(u)=m_1$.*

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, m_4)}$ such that

$$\gamma(u_i) \le m_4 \text{ for } i = 1,2,3, \quad m_2 < \alpha(u_1);$$

$$m_1 < \beta(u_2) \text{ with } \alpha(u_2) < m_2; \ \beta(u_3) < m_1.$$

3. MAIN RESULTS

Let E be the Banach space $C^1[0,1]$ with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u'\|_0 = \max_{0\le t\le 1}|u'(t)|$. Set

$$K = \Big\{u \in E : u \text{ is nonnegative, concave on } [0,1] \text{ and } u \text{ satisfies (1.2)}, \tag{3.1}$$
$$\min_{t\in[\sigma,1-\sigma]} u(t) \ge \tau_1 \|u\|_0\Big\}.$$

Clearly, K is a cone of E. Now from Lemma 2.1, the problem (1.1) and (1.2) has a solution u if and only if u is the fixed point of the operator equation

$$u(t) = \int_0^1 G(t,s)\,h(s)\,f(s,u(s),u'(s))\,ds + x(t)\,A(hf) + y(t)\,B(hf) := \ (Tu)(t).$$

Assume that (A_1), (A_2) and (A_3) hold. By Lemma 2.2, we know that $Tu(t) \ge 0$ and $(Tu)''(t) = -h(t)f(t,u(t),u'(t)) \le 0$ for $u \in K$. Moreover, according to Lemma 2.2, we can conclude that $Tu \in K$. Applying Arzela-Ascoli lemma, it is easy to see that T is completely continuous.

Now we give a lemma which is important in establishing the existence of triple positive solutions of the problem (1.1) and (1.2).

For notational convenience, we denote

$$\Lambda_1 = \frac{b + \sum_{i=1}^{m-2} a_i\xi_i}{a - \sum_{i=1}^{m-2} a_i}, \quad \Lambda_2 = \frac{d + \sum_{i=1}^{m-2} b_i(1-\xi_i)}{c - \sum_{i=1}^{m-2} b_i}.$$

Lemma 3.1. *Assume (A_1) holds, if $u \in K$, then*

$$\|u\|_0 \le \tau_2 \|u'\|_0, \tag{3.2}$$

where

$$\tau_2 = \max\{\tau_{21}, \ \tau_{22}\},$$

$$\tau_{21} = \max\left\{\frac{a\xi_1 + b}{a - \sum_{i=1}^{m-2} a_i}, \Lambda_1\left(1 + \frac{\xi_{m-2}(a-a_1)}{a_1\xi_1}\right), \Lambda_1\left(1 + \min_{1\le i\le m-2}\frac{a-a_i}{a_i\xi_i}\right)\right\},$$

$$\tau_{22} = \max\left\{\min_{1\le i\le m-2}\frac{c\Lambda_2 - d}{b_i(1-\xi_i)}, \frac{(1-\xi_1)(c\Lambda_2 - d)}{b_{m-2}(1-\xi_{m-2})}, \frac{d + c - c\xi_{m-2}}{c - \sum_{i=1}^{m-2} b_i}\right\}.$$

Proof. For $u \in K$, we suppose $\|u\|_0 = \max_{0\le t\le 1} u(t) = u(\xi)$. If $\xi = 0$, then by the concavity of u and the condition (A_1), we know that $u'(0) \le 0$ and

$$au(0) - bu'(0) \ge au(0) > \sum_{i=1}^{m-2} a_i u(0) \ge \sum_{i=1}^{m-2} a_i u(\xi_i),$$

which contradicts the assumption that u satisfies (1.2). Similarly, if $\xi = 1$, then we can get that

$$cu(1) + du'(1) \ge cu(1) > \sum_{i=1}^{m-2} b_i u(1) \ge \sum_{i=1}^{m-2} b_i u(\xi_i),$$

which is a contradiction with the assumption that u satisfies (1.2). Thus $\|u\|_0 = u(\xi), \xi \in (0,1)$. Furthermore

$$\|u'\|_0 = \max_{0\le t\le 1} |u'(t)| = \max\{|u'(0)|, |u'(1)|\}.$$

In the following, we concentrate on the existence of constant τ_2.

First, we suppose that $\|u'\|_0 = |u'(0)| = u'(0)$.

By the concavity of u on $[0,1]$, we have $u'(0) \ge (u(\xi_i) - u(0))/\xi_i$, $i = 1, 2, \ldots, m-2$. Take into account that $au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, it follows that

$$\sum_{i=1}^{m-2} a_i \xi_i u'(0) \ge \sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i u(0) = au(0) - bu'(0) - \sum_{i=1}^{m-2} a_i u(0),$$

hence

$$u(0) \le \frac{b + \sum_{i=1}^{m-2} a_i \xi_i}{a - \sum_{i=1}^{m-2} a_i} u'(0) = \Lambda_1 u'(0). \tag{3.3}$$

From $\|u\|_0 = u(\xi)$, $\xi \in (0,1)$, we know that there are three cases to be considered.

Case 1. $\xi \in (0, \xi_1]$. The concavity of u implies $u'(0) \ge (u(\xi) - u(0))/\xi$, so

$$au(\xi) - au(0) \le a\xi u'(0). \tag{3.4}$$

Since $u(\xi_i) \le u(\xi)$, we have

$$au(0) - bu'(0) \le \sum_{i=1}^{m-2} a_i u(\xi). \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$u(\xi) \le \frac{a\xi + b}{a - \sum_{i=1}^{m-2} a_i} u'(0) \le \frac{a\xi_1 + b}{a - \sum_{i=1}^{m-2} a_i} u'(0),$$

that is

$$\|u\|_0 \le \frac{a\xi_1 + b}{a - \sum_{i=1}^{m-2} a_i} \|u'\|_0. \tag{3.6}$$

Case 2. $\xi \in (\xi_1, \xi_{m-2}]$. According to the property of b, we get

$$au(0) = bu'(0) + \sum_{i=1}^{m-2} a_i u(\xi_i) \geq a_1 u(\xi_1),$$

which combines with the inequality $\frac{u(\xi_1)-u(0)}{\xi_1} \geq \frac{u(\xi)-u(0)}{\xi}$ and (3.3), we have

$$u(\xi) \leq \left(1 + \frac{\xi(a-a_1)}{a_1\xi_1}\right) u(0) \leq \Lambda_1 \left(1 + \frac{\xi_{m-2}(a-a_1)}{a_1\xi_1}\right) u'(0),$$

that is

$$\|u\|_0 \leq \Lambda_1 \left(1 + \frac{\xi_{m-2}(a-a_1)}{a_1\xi_1}\right) \|u'\|_0 . \tag{3.7}$$

Case 3. $\xi \in (\xi_{m-2}, 1)$. There are $au(0) - bu'(0) \geq a_i u(\xi_i), i = 1, 2, \ldots, m-2$, so

$$\frac{\frac{a}{a_i}u(0) - u(0)}{\xi_i} \geq \frac{\frac{a}{a_i}u(0) - \frac{b}{a_i}u'(0) - u(0)}{\xi_i} \geq \frac{u(\xi_i) - u(0)}{\xi_i} \geq \frac{u(\xi) - u(0)}{\xi}.$$

It follows that

$$u(\xi) \leq \min_{1\leq i\leq m-2}\left(1 + \frac{\xi(a-a_i)}{a_i\xi_i}\right) u(0) \leq \min_{1\leq i\leq m-2}\left(1 + \frac{a-a_i}{a_i\xi_i}\right) u(0),$$

combining with (3.3), we obtain that

$$\|u\|_0 = u(\xi) \leq \Lambda_1 \left(1 + \min_{1\leq i\leq m-2} \frac{a-a_i}{a_i\xi_i}\right) \|u'\|_0 . \tag{3.8}$$

Consequently, from (3.6), (3.7) and (3.8), we have

$$\|u\|_0 \leq \tau_{21} \|u'\|_0 . \tag{3.9}$$

Secondly, suppose that $\|u'\|_0 = |u'(1)|$. Again, by the concavity of u on $[0,1]$, we have

$$u'(1) \leq \frac{u(\xi_i) - u(1)}{\xi_i - 1}, \; i = 1, 2, \ldots, m-2,$$

and by the condition $cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, we know that

$$-\sum_{i=1}^{m-2} b_i (1-\xi_i) u'(1) \geq \sum_{i=1}^{m-2} b_i u(\xi_i) - \sum_{i=1}^{m-2} b_i u(1) = cu(1) + du'(1) - \sum_{i=1}^{m-2} b_i u(1),$$

therefore

$$u(1) \leq \frac{d + \sum_{i=1}^{m-2} b_i (1-\xi_i)}{c - \sum_{i=1}^{m-2} b_i} (-u'(1)) = \Lambda_2 |u'(1)| . \tag{3.10}$$

Similarly, we have the following discussion.

Case 1. $\xi \in (0, \xi_1]$. There are $\frac{u(\xi)-u(1)}{\xi-1} \geq \frac{u(\xi_i)-u(1)}{\xi_i-1}$ for $i = 1, 2, \ldots, m-2$, so we get

$$cu(1) + du'(1) \geq b_i u(\xi_i) \geq \frac{b_i(\xi_i - \xi)}{1-\xi} u(1) + \frac{b_i(1-\xi_i)}{1-\xi} u(\xi),$$

thus

$$\frac{b_i(1-\xi_i)}{1-\xi} u(\xi) \leq cu(1) + du'(1).$$

By (A_1), it is easy to check that $c\Lambda_2 - d > 0$, and in view of (3.10), we have

$$u(\xi) \le \frac{c\Lambda_2 - d}{b_i(1-\xi_i)}(-u'(1)) \text{ for } i = 1,\ 2,\ \ldots,\ m-2.$$

That is

$$\|u\|_0 \le \min_{1\le i\le m-2}\left\{\frac{c\Lambda_2 - d}{b_i(1-\xi_i)}\right\}\|u\|_0 . \tag{3.11}$$

Case 2. $\xi \in (\xi_1, \xi_{m-2}]$. The concavity of u implies that $\frac{u(\xi_{m-2})-u(1)}{\xi_{m-2}-1} \le \frac{u(\xi)-u(1)}{\xi-1}$. So

$$cu(1) + du'(1) \ge b_{m-2}u(\xi_{m-2}) \ge b_{m-2}\left(\frac{1-\xi_{m-2}}{1-\xi}u(\xi) + \frac{\xi_{m-2}-\xi}{1-\xi}u(1)\right),$$

by (3.10) and $c\Lambda_2 - d > 0$, it follows that

$$u(\xi) \le \frac{(1-\xi)(c\Lambda_2 - d)}{b_{m-2}(1-\xi_{m-2})}(-u'(1)) \le \frac{(1-\xi_1)(c\Lambda_2 - d)}{b_{m-2}(1-\xi_{m-2})}(-u'(1)),$$

that is

$$\|u\|_0 \le \frac{(1-\xi_1)(c\Lambda_2 - d)}{b_{m-2}(1-\xi_{m-2})}\|u'\|_0 . \tag{3.12}$$

Case 3. $\xi \in (\xi_{m-2}, 1)$. Again, by the concavity of u we have $u'(1) \le \frac{u(\xi)-u(1)}{\xi-1}$, thus

$$cu(\xi) + cu(1-\xi)u'(1) \le cu(1).$$

In view of $u \in K$, we get $cu(1) + du'(1) \le \sum_{i=1}^{m-2} b_i u(\xi)$, and

$$cu(\xi) + cu(1-\xi)u'(1) + du'(1) \le \sum_{i=1}^{m-2} b_i u(\xi),$$

therefore

$$\|u\|_0 = u(\xi) \le \frac{d + c(1-\xi)}{c - \sum_{i=1}^{m-2} b_i}(-u'(1)) \le \frac{d + c - c\xi_{m-2}}{c - \sum_{i=1}^{m-2} b_i}\|u'\|_0 . \tag{3.13}$$

By (3.11), (3.12) and (3.13), we obtain

$$\|u\|_0 \le \tau_{22}\|u'\|_0 . \tag{3.14}$$

So from (3.9) and (3.14), we get $\|u\|_0 \le \max\{\tau_{21}, \tau_{22}\}\|u'\|_0 = \tau_2\|u'\|_0$. □

We are now ready to apply Avery-Peterson's fixed point theorem to the operator T to give the sufficient conditions for the existence of at least three positive solutions to the problem (1.1) and (1.2).

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals β and γ, and the nonnegative continuous functional θ be defined on the cone K by

$$\alpha(u) = \min_{\sigma\le t\le 1-\sigma} u(t), \quad \gamma(u) = \max_{0\le t\le 1}|u'(t)|$$

$$\theta(u) = \beta(u) = \max_{0\le t\le 1} u(t) \text{ for } u \in K.$$

Now for convenience we introduce the following notations. Let

$$\begin{aligned} S &= \max\left\{\left|x'(0)\int_0^1 \frac{1}{\rho} y(s)\,h(s)\,ds + x'(0)\,A(h) + y'(0)\,B(h)\right|,\right. \\ &\qquad \left.\left|y'(1)\int_0^1 \frac{1}{\rho} x(s)\,h(s)\,ds + x'(1)\,A(h) + y'(1)\,B(h)\right|\right\}, \end{aligned}$$

$$\begin{aligned} M &= \min\left\{\int_0^1 G(\sigma,s)\,h(s)\,ds + x(\sigma)\,A(h) + y(\sigma)\,B(h),\right. \\ &\qquad \left.\int_0^1 G(1-\sigma,s)\,h(s)\,ds + x(1-\sigma)\,A(h) + y(1-\sigma)\,B(h)\right\} \end{aligned}$$

and

$$N = \max_{0\le t\le 1}\left(\int_0^1 G(t,s)\,h(s)\,ds + x(t)A(h) + y(t)B(h)\right).$$

Theorem 3.2. *Suppose* $(A_1)-(A_3)$ *hold and* $f(t,0,0)\not\equiv 0$ *for* $t\in[0,1]$. *If there exist positive numbers* m_1, m_2 *and* m_4 *with* $m_1 < m_2$ *such that the following conditions are satisfied:*

(C_1) $f(t,\mu,\nu) \le m_4/S$ *for* $(t,\mu,\nu)\in[0,1]\times[0,\tau_2 m_4]\times[-m_4,m_4]$;

(C_2) $f(t,\mu,\nu) > m_2/M$ *for* $(t,\mu,\nu)\in[\sigma,1-\sigma]\times[m_2,m_2/\tau_1]\times[-m_4,m_4]$;

(C_3) $f(t,\mu,\nu) \le m_1/N$ *for* $(t,\mu,\nu)\in[0,1]\times[0,m_1]\times[-m_4,m_4]$.

Then the problem (1.1) and (1.2) has at least three positive solutions u_1, u_2, *and* u_3 *satisfying*

$$\max_{0\le t\le 1}|u_i'(t)| \le m_4 \text{ for } i=1,2,3;\quad m_2 < \min_{\sigma\le t\le 1-\sigma} u_1(t); \tag{3.15}$$

$$m_1 < \max_{0\le t\le 1} u_2(t) \text{ with } \min_{\sigma\le t\le 1-\sigma} u_2(t) < m_2;\ \max_{0\le t\le 1} u_3(t) < m_1. \tag{3.16}$$

Proof. By the definition of operator T and its properties, it suffices to show that the conditions of Lemma 2.3 hold with respect to T.

We first show that if (C_1) is satisfied, then

$$T:\overline{P(\gamma,m_4)} \to \overline{P(\gamma,\ m_4)}. \tag{3.17}$$

In fact, for $u\in\overline{P(\gamma,m_4)}$, there is $\gamma(u) = \max_{0\le t\le 1}|u'(t)| \le m_4$. With Lemma 3.1, there is $\|u\|_0 \le \tau_2\|u'\|_0 \le \tau_2 m_4$, and assumption (C1) implies $f(t,u(t),u'(t)) \le m_4/S$ for $t\in[0,1]$. On the other hand, for $u\in K$, there is $Tu\in K$, then Tu

is concave on $[0,1]$, and $\max_{0\le t\le 1}\left|(Tu)'(t)\right| = \max\left\{\left|(Tu)'(0)\right|, \left|(Tu)'(1)\right|\right\}$, so

$$\begin{aligned}
\gamma(Tu) &= \max_{0\le t\le 1}\left|(Tu)'(t)\right| \\
&= \max_{0\le t\le 1}\left|y'(t)\int_0^t \frac{1}{\rho}x(s)h(s)f(s,u(s),u'(s))ds\right. \\
&\quad \left.+x'(t)\int_t^1 \frac{1}{\rho}y(s)h(s)f(s,u(s),u'(s))ds + x'(t)A(hf) + y'(t)B(hf)\right| \\
&\le \frac{m_4}{S}\max\left\{\left|x'(0)\int_0^1 \frac{1}{\rho}y(s)h(s)ds + x'(0)A(h) + y'(0)B(h)\right|,\right. \\
&\quad \left.\left|y'(1)\int_0^1 \frac{1}{\rho}x(s)h(s)ds + x(1)A(h) + y'(1)B(h)\right|\right\} = m_4.
\end{aligned}$$

Therefore, (3.17) is satisfied.

We choose $u(t) = m_2/\tau_1$ for $0\le t\le 1$. It is easy to see that

$$u(t) = m_2/\tau_1 \in P(\gamma,\theta,\alpha,m_2,m_2/\tau_1,m_4) \text{ and } \alpha(u) > m_2.$$

Hence

$$\{P(\gamma,\theta,\alpha,m_2,m_2/\tau_1,m_4) : \alpha(u) > m_2\} \ne \emptyset.$$

For $u \in P(\gamma,\theta,\alpha,m_2,m_2/\tau_1,m_4)$, there is $m_2 \le u(s) \le m_2/\tau_1$ and $|u'(s)| \le m_4$ for $s\in[\sigma,1-\sigma]$. Hence by condition (C_2), one has that $f(t,u(t),u'(t)) > m_2/M$ for $t\in[\sigma,1-\sigma]$. So by the definition of the functional α we see that

$$\begin{aligned}
\alpha(Tu) &= \min_{\sigma\le t\le 1-\sigma} Tu(t) = \min\{(Tu)(\sigma), (Tu)(1-\sigma)\} \\
&= \min\left\{\int_0^1 G(\sigma,s)h(s)f(s,u(s),u'(s))ds + x(\sigma)A(hf) + y(\sigma)B(hf),\right. \\
&\quad \left.\int_0^1 G(1-\sigma,s)h(s)f(s,u(s),u'(s))ds + x(1-\sigma)A(hf) + y(1-\sigma)B(hf)\right\} \\
&\ge \frac{m_2}{M}\min\left\{\int_0^1 G(\sigma,s)h(s)ds + x(\sigma)A(h) + y(\sigma)B(h),\right. \\
&\quad \left.\int_0^1 G(1-\sigma,s)h(s)ds + x(1-\sigma)A(h) + y(1-\sigma)B(h)\right\} = m_2.
\end{aligned}$$

Therefore, we get $\alpha(Tu) > b$ for $u \in P(\gamma,\theta,\alpha,m_2,m_2/\tau_1,m_4)$ and condition (B_1) in Lemma 2.3 is satisfied.

We now prove that (B_2) in Lemma 2.3 holds. In fact, if $u\in P(\gamma,\ \theta,\ m_2,\ m_4)$ with $\theta(Tu) > m_2/\tau_1$, then

$$\alpha(Tu) = \min_{\sigma\le t\le 1-\sigma} Tu(t) \ge \tau_1 \max_{0\le t\le 1} Tu(t) = \tau_1\theta(Tu) > m_2.$$

Finally, we assert that (B_3) in Lemma 2.3 also holds.

Since $\beta(0) = 0 < m_1$, so $0 \notin Q(\gamma, \beta, m_1, m_4)$. Assume that $u \in Q(\gamma, \beta, m_1, m_4)$ with $\beta(u) = m_1$, then, by the condition (C_3) we obtain that

$$\begin{aligned}\beta(Tu) &= \max_{0\le t\le 1} Tu(t)\\ &= \max_{0\le t\le 1} \int_0^1 G(t,s)\,h(s)\,f(s,u(s),u'(s))ds + x(t)A(hf) + y(t)B(hf)\\ &\le \frac{m_1}{N}\max_{0\le t\le 1}\left[\int_0^1 G(t,s)\,h(s)\,ds + x(t)\,A(h) + y(t)\,B(h)\right] = m_1.\end{aligned}$$

To sum up, $(B_1)-(B_3)$ hold. Thus from 2.3 and the assumption that $f(t,0,0) \not\equiv 0$ on $[0,1]$, the BVP (1.1) and (1.2) has at least three positive solutions u_1, u_2, u_3 such that (3.15) and (3.16) hold. The proof is complete. □

4. **EXAMPLE**

In this section, we give an example to illustrate our results.

Let $h(t) = 1$ and $m = 4$, $a = c = 4$, $b = d = 2$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $a_1 = a_2 = b_1 = b_2 = 1/2$. We consider the following BVP

$$u''(t) + f(t, u(t), u'(t)) = 0,\ 0 < t < 1 \tag{4.1}$$

$$4u(0) - 2u'(0) = u(1/4) + u(1/2),\ 4u(1) + 2u'(1) = u(1/4) + u(1/2), \tag{4.2}$$

where

$$f(t,\mu,\nu) = \begin{cases} \frac{1}{2}t + \frac{7}{10}\mu^3 + \left(\frac{\nu}{60}\right)^3,\ t\in[0,1], \mu\in(-\infty,4], \nu\in(-\infty,\infty);\\ \frac{1}{2}t + \frac{7}{10}(5-\mu)\mu^3 + \left(\frac{\nu}{60}\right)^3,\ t\in[0,1], \mu\in(4,5), \nu\in(-\infty,\infty);\\ \frac{1}{2}t + \frac{7}{10}(\mu-5)\mu^3 + \left(\frac{\nu}{60}\right)^3,\ t\in[0,1], \mu\in(5,5.5], \nu\in(-\infty,\infty);\\ \frac{1}{2}t + \frac{9317}{160} + \left(\frac{\nu}{60}\right)^3,\ t\in[0,1], \mu\in(5.5,\infty), \nu\in(-\infty,\infty).\end{cases}$$

It is easy to see that $x(t) = 4t + 2$, $y(t) = -4t + 6$ and the conditions $(A_1) - (A_3)$ hold and $f(t,0,0) \not\equiv 0$ on $[0,1]$. By some calculations, we have $\rho = 32$, $\Delta = -512$, $\tau_1 = 1/2$, $\tau_2 = 3/2$ and $S = 1/2$, $M = 45/64$, $N = 47/64$. If we choose $\sigma = 1/4$, $m_1 = 1$, $m_2 = 2$ and $m_4 = 30$, then $f(t,\mu,\nu)$ satisfies

$$f(t,\mu,\nu) \le 60 = m_4/S \quad \text{for } (t,\mu,\nu) \in [0,1]\times[0,45]\times[-30,30];$$

$$f(t,\mu,\nu) > 128/45 = m_2/M \quad \text{for } (t,\mu,\nu) \in [1/4,3/4]\times[2,4]\times[-30,30];$$

$$f(t,\mu,\nu) \le 64/47 = m_1/N \quad \text{for } (t,\mu,\nu) \in [0,1]\times[0,1]\times[-30,30].$$

Then all assumptions of Theorem 3.2 hold. Thus by Theorem 3.2, the problem (4.1) and (4.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{0\le t\le 1}|u_i'(t)| \le 30 \quad \text{for } i = 1,2,3; \quad 2 < \min_{1/4\le t\le 3/4} u_1(t);$$

$$1 < \max_{0\le t\le 1} u_2(t) \text{ with } \min_{1/4\le t\le 3/4} u_2(t) < 2; \quad \max_{0\le t\le 1} u_3(t) < 1.$$

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