Dynamic Systems and Applications 17 (2008) 201-220

L^2-STABILITY OF VECTOR EQUATIONS WITH NONLINEAR CAUSAL MAPPINGS

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ABSTRACT. Nonlinear vector equations with causal mappings are considered. These equations include differential, difference, differential-delay, integro-differential and other traditional equations. Estimates for the L^2-norm of solutions are established. The obtained estimates give us explicit conditions for the L^2-stability, absolute stability and input-to-state stability of the considered equations as well as bounds for the regions of attraction of stationary states. The suggested approach enables us to consider various classes of equations from the unified point of view.

Key words: causal mappings, differential equations, differential delay equations, difference equations with continuous time, L^2-stability, absolute stability, input-to-state stability

AMS (MOS) subject classification: 34K20, 34K99, 93D05, 93D25

1. INTRODUCTION AND MAIN DEFINITIONS

The present paper is devoted to equations in a Euclidean space with nonlinear causal mappings (operators). These equations include differential, difference, differential-delay, integro-differential and other traditional equations. For the details see the excellent books [4, 23]. The stability theory of nonlinear equations with causal mappings is in an early stage of development. The basic method for the stability analysis is the direct Liapunov method. But finding the Liapunov functionals for equations with causal mappings is a difficult mathematical problem.

Below we establish explicit conditions that provide the L^2-stability, absolute stability and input-to-state stability for a class of equations with causal mappings. To the best of our knowledge these stabilities for equations with causal mappings were not explored in the available literature. As it is shown below, in appropriate situations our results are exact. The literature on stability of continuous systems is very rich, cf. [17, 20, 21, 24] and references therein. The classical results were developed in the interesting papers [26], [1], [14, 18], [22].

The basic stability results for differential-delay equations are presented in the well-known books [12, 13, 20]. Recently stability of nonlinear retarded systems has

This research was supported by the Kamea fund of the Israel.

Received December 01, 2006

been investigated by many authors, cf. [16, 11, 25], etc. Mainly the direct Liapunov method and frequency criteria are applied.

A deep investigation of linear causal operators is presented in the book [15]. The papers [2, 5] also should be mentioned. In the paper [5], the existence and uniqueness of local and global solutions to the Cauchy problem for equations with causal operators in a Banach space are established. In the paper [2] it is proved that the input-output stability of vector equations with causal operators is equivalent to the causal invertibility of causal operators.

The present paper is organized as follows. It consists of 10 sections. In this section we define the causal mappings and consider an example of a causal mapping. In Section 2 preliminary results are collected. The main result-Theorem 3.1 on solution estimates for the considered equations is proved in Section 3. Equations with differential linear parts are investigated in Sections 4 and 5. Section 6 is devoted to difference equations with continuous time and causal nonlinearities. Equations with differential-delay linear parts are explored in Sections 6-10.

Let $\mathbb{C}^n$ be a Euclidean space with the Euclidean norm $\|\cdot\|_n$. By I we denote the unit operator in the corresponding space.

Furthermore, for a positive $T \leq \infty$, let E be a Banach space of functions defined on $[0,T]$ with values in $\mathbb{C}^n$. For all $\tau \in [0,T)$ and $w \in E$, let the projections P_τ be defined by

$$(P_\tau w)(t) = \begin{cases} w(t) & \text{if } 0 \leq t \leq \tau, \\ 0 & \text{if } \tau < t \leq T \end{cases} . \tag{1.1}$$

In addition, $P_T w = w$.

Definition 1.1. A mapping $F : E \to E$ satisfying the condition

$$P_\tau F P_\tau = P_\tau F \quad (\tau \in [0,T]) \tag{1.2}$$

will be called a causal mapping (operator).

This definition is somewhat different from the definition of the causal operator suggested in [4], in the linear case our definition coincides with the one accepted in [6].

Let us point an example of a causal mapping. To this end denote by $L^p(\omega) = L^p(\omega, \mathbb{C}^n)$ $(1 < p < \infty)$ the space of functions defined on a set $\omega \subset \mathbb{R}$ with values in $\mathbb{C}^n$ and the finite norm

$$|w|_{L^p(\omega)} = [\int_\omega \|w(t)\|_n^p dt]^{1/p} \ (w \in L^p(\omega)).$$

In addition,

$$|w|_{C(\omega)} = \sup_{t\in\omega} \|w(t)\|_n$$

for a vector valued function w defined and bounded on ω.

Consider in $L^2(0,T)$ the operator

$$(Fw)(t) = f(t, w(t)) + \int_0^t k(t, s, w(s))ds \ \ (w \in L^2(0,T))$$

with a continuous function $k : [0,T]^2 \times \mathbb{C}^n \to \mathbb{C}^n$ and a continuous function $f : [0,T] \times \mathbb{C}^n \to \mathbb{C}^n$. For each $\tau \in [0,T)$ we have

$$(P_\tau F w)(t) = f_\tau(t, w(t)) + \int_0^\tau k_\tau(t, s, w(s))ds$$

where

$$k_\tau(t, s, w(s)) = \begin{cases} k(t, s, w(s)) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}$$

$(0 \le s \le t)$, and

$$f_\tau(t, w(t)) = \begin{cases} f(t, w(t)) & \text{if } 0 \le t \le \tau, \\ 0 & \text{if } \tau < t \le T \end{cases}.$$

Thus (1.2) holds and the considered mapping is causal. Note that, the integral operator

$$\int_0^c k(t, s, w(s))ds$$

with a fixed positive $c \le T$ is not causal.

2. PRELIMINARY RESULTS

Put $R_+ = [0, \infty)$. *Everywhere below F is a continuous causal mapping acting in $L^2(R_+)$.* Consider the equation

$$x(t) = f(t) + \int_0^t Q_0(t, t_1)(Fx)(t_1)dt_1 \ \ (t > 0), \tag{2.1}$$

where Q_0 is a measurable $n \times n$-matrix kernel defined for $0 \le s \le t \le \infty$, and $f \in L^2(R_+)$ is given.

Definition 2.1. A solution of (2.1) is a vector-valued function x defined on R_+ which is in $L^2(0,\tau)$ for any finite $\tau > 0$ and satisfies (2.1) a.e. on R_+.

It is assumed that there is a constant q, such that

$$|Fw|_{L^2(R_+)} \le q \, |w|_{L^2(R_+)} \ (w \in L^2(R_+)). \tag{2.2}$$

Lemma 2.2. *Let $F : L^2(R_+) \to L^2(R_+)$ be a continuous causal mapping satisfying condition (2.2). Then for all $\tau > 0$ and $w \in L^2(0,\tau)$, we have the inequality*

$$|Fw|_{L^2(0,\tau)} \le q |Fw|_{L^2(0,\tau)}.$$

Proof. From (1.2) it follows that

$$|Fw|_{L^2(0,\tau)} = |P_\tau Fw|_{L^2(R_+)} = |P_\tau FP_\tau w|_{L^2(R_+)} \leq$$
$$|FP_\tau w|_{L^2(R_+)} \leq q\ |P_\tau w|_{L^2(R_+)} = q|w|_{L^2(0,\tau)},$$

as claimed. □

In $L^2(R_+)$ introduce the operator V_0 by

$$(V_0 w)(t) = \int_0^t Q_0(t,t_1)w(t_1)dt_1 \ \ (t > 0;\ w \in L^2(R_+)).$$

Lemma 2.3. *Let V_0 be compact in $L^2(0,\tau)$ for each finite τ, and the conditions (2.2), and*

$$q|V_0|_{L^2(R_+)} < 1 \tag{2.3}$$

hold. Then (2.1) has at least one solution. Moreover any solution x of (2.1) satisfies the inequality

$$|x|_{L^2(R_+)} \leq \frac{|f|_{L^2(R_+)}}{1 - q|V_0|_{L^2(R_+)}}. \tag{2.4}$$

Proof. On $L^2(0,T)$, $T < \infty$, let us define the mapping Φ by $(\Phi w)(t) = f(t) + (V_0 Fw)(t)$ for a $w \in L^2(0,T)$. Hence, according to the previous lemma, for an $r > 0$, large enough,

$$|\Phi w|_{L^2(0,T)} \leq |f|_{L^2(0,T)} + |V_0|_{L^2(0,T)} q |w|_{L^2(0,T)} \leq r \ \ (|w|_{L^2(0,T)} \leq r).$$

So Φ maps a bounded set of $L^2(0,T)$ into itself. Now the existence of a solution $x(t)$ is due to the Shauder Fixed Point Theorem, since V_0 is compact. Furthermore, from (2.1) it follows

$$|x|_{L^2(0,T)} \leq |f|_{L^2(0,T)} + |V_0|_{L^2(0,T)} q |x|_{L^2(0,T)} \ \ (T \leq \infty).$$

Now (2.3) implies the required result. □

Furthermore, let $Q_0(t,s) = Q(t-s)$ with $Q \in L^1(R_+)$. Then $V_0 = V_Q$, where the operator V_Q is defined by

$$(V_Q w)(t) = \int_0^t Q(t-t_1)w(t_1)dt_1 \ \ (t > 0).$$

Let

$$\tilde{Q}(z) := \int_0^\infty e^{-zt} Q(t)dt \ \ (Re\ z \geq 0)$$

be the Laplace transform of Q. Then by the Parseval equality we easily get $|V_Q|_{L^2} = \Lambda_Q$ where

$$\Lambda_Q := \sup_{s \in \mathbb{R}} \|\tilde{Q}(is)\|_n.$$

So the previous lemma implies

Corollary 2.4. *Let $Q \in L^1(R_+)$ and the conditions (2.2) and*

$$q\Lambda_Q < 1 \tag{2.5}$$

hold. Then the equation

$$x(t) = f(t) + \int_0^t Q(t-t_1)(Fx)(t_1)dt_1 \ \ (t > 0), \tag{2.6}$$

has at least one solution. Moreover any its solution x, satisfies the inequality

$$|x|_{L^2(R_+)} \le \frac{|f|_{L^2(R_+)}}{1 - q\Lambda_Q}. \tag{2.7}$$

3. THE MAIN RESULT

For a positive $r \le \infty$ denote $\Omega(r) = \{v \in L^2(R_+) : \|w(t)\|_n \le r,\ t \ge 0\}$. Let $F : \Omega(r) \to L^2(R_+)$ be a continuous causal mapping. Again consider equation (2.1). It is assumed that there is a constant $q = q(r)$, such that

$$|Fv|_{L^2(R_+)} \le q\ |v|_{L^2(R_+)}\ (v \in \Omega(r)). \tag{3.1}$$

Theorem 3.1. *Let V_0 be compact in $L^2(0, \tau)$ for each finite τ, and the conditions (3.1), and (2.3) hold. In addition, let a solution x of equation (2.1) (if it exists) satisfy the a priory estimate*

$$\|x(t)\|_n < r,\ t \ge 0. \tag{3.2}$$

Then (2.1) really has at least one solution $x \in \Omega(r)$. Moreover inequality (2.4) is valid.

Proof. If $r = \infty$, then the required result is due to Lemma 2.1. Now let $r < \infty$. Define on $L^2(0, T)$ the function

$$h(w) = \begin{cases} 1, & w \in \Omega(r), \\ 0, & w \notin \Omega(r) \end{cases}.$$

Such a function always exists due to the Urysohn theorem [3, p. 15]. In addition put $\tilde{F} = hF$. According to (3.1), the inequality

$$|\tilde{F}w|_{L^2(R_+)} \le q|w|_{L^2(R_+)} \quad (w \in L^2(R_+))$$

is valid. Due to Lemma 2.2 a solution $\tilde{x}$ of (2.1) with $F = \tilde{F}$ exists and satisfies inequality (2.4). This and (3.2) prove the theorem. □

The previous theorem and Corollary 2.4 imply

Corollary 3.2. *Let $Q \in L^1(R_+)$ and conditions (3.1), (3.2) and (2.5) hold. Then equation (2.6) has at least one solution $x \in \Omega(r)$. Moreover any its solution x, satisfies inequality (2.7).*

Let the conditions (3.1), (3.2),

$$f \in \Omega(r) \text{ and } \gamma_0 := \sup_{t\geq 0}[\int_0^t Q_0^2(t,s)ds\,]^{1/2} < \infty \tag{3.3}$$

hold. Then according to (2.1) and the Schwarz inequality,

$$|x|_{C(R_+)} \leq |f|_{C(R_+)} + \gamma_0 q|x|_{L^2(R_+)}.$$

Now under (2.3), inequality (2.4) implies

$$|x|_{C(R_+)} \leq \zeta_C(f,Q_0) := |f|_{C(R_+)} + \frac{\gamma_0 q|f|_{L^2(R_+)}}{1-|V_0|_{L^2(R_+)}q}. \tag{3.4}$$

Thus if

$$\zeta(f,Q_0) < r \tag{3.5}$$

then (3.2) holds and thanks to the previous theorem we arrive at the following result.

Corollary 3.3. *Let V_0 be compact in $L^2(0,\tau)$ for each finite τ, and the conditions (3.1), (3.3), (3.5) and (2.3) hold. Then (2.1) has at least one solution $x \in \Omega(r)$. Moreover inequalities (2.4) and (3.4) are valid.*

In particular, if $Q_0(t,s) = Q(t-s)$ and

$$Q \in L^2(R_+) \cap L^1(R_+), \tag{3.6}$$

then $\gamma_0 = |Q|_{L^2(R_+)}$ and $\zeta_C(f,Q_0) = \zeta_C(f,Q)$ where

$$\zeta_C(f,Q) := |f|_{C(R_+)} + \frac{q|Q|_{L^2(R_+)}|f|_{L^2(R_+)}}{1-\Lambda_Q q}.$$

Now Theorem 3.1 yields

Corollary 3.4. *Let the conditions (3.6), (3.1), (2.5) and $\zeta_C(f,Q) < r$ hold. Then equation (2.1) has at least one solution $x \in \Omega(r)$. Moreover the inequalities (2.7) and $|x|_{C(R_+)} \leq \zeta_C(f,Q)$ are valid.*

4. EQUATIONS WITH DIFFERENTIAL LINEAR PARTS

Consider the equation

$$\dot{x}(t) = Ax(t) + [Fx](t) \quad (\dot{x} = dx/dt), \tag{4.1}$$

where A is a constant Hurwitzian $n \times n$-matrix. Take the initial condition

$$x(0) = x_0 \in \mathbb{C}^n. \tag{4.2}$$

From (4.1) we get the equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-t_1)}(Fx)(t_1)dt_1 \quad (t > 0). \tag{4.3}$$

A continuous solution of the latter integral equation will be called *a mild solution of problem (4.1), (4.2)*. Put

$$\Lambda_A := \sup_{s\in\mathbb{R}} \|(A - isI)^{-1}\|_n.$$

Thanks to Corollary 2.4, under the conditions (2.2) and

$$q\Lambda_A < 1, \tag{4.4}$$

problem (4.1), (4.2) has at least one mild solution. Moreover any its mild solution x, satisfies the inequality

$$|x|_{L^2(R_+)} \le \frac{|e^{At}x_0|_{L^2(R_+)}}{1 - q\Lambda_A}. \tag{4.5}$$

From (4.3) and the Schwarz inequality we get

$$|x|_{C(R_+)} \le \zeta(x_0, q, A) := (|e^{At}|_{C(R_+)} + \frac{q|e^{At}|^2_{L^2(R_+)}}{1 - q\Lambda_A})\|x_0\|_n. \tag{4.6}$$

So Corollary 3.4 implies

Theorem 4.1. *Let the conditions (3.1), (4.4) and $\zeta(x_0, q, A) < r$ hold. Then problem (4.1), (4.2) has at least one mild solution. Moreover any mild solution x of (4.1), (4.2) satisfies inequalities (4.5) and (4.6).*

Let us derive bounds for Λ_A and $\zeta(x_0, q, A)$. Let $\lambda_k(A)$ $(k = 1, \dots, n)$ denote the eigenvalues of matrix A counting with their multiplicities, and

$$g(A) := (N^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2)^{1/2},$$

where $N(A)$ is the Hilbert-Schmidt (Frobenius) norm of A, i.e. $N^2(A) = Trace(AA^*)$. The following properties of $g(A)$ are valid [8, Section 2.1]:

$$g(A) \le \sqrt{1/2}N(A^* - A) \text{ and } g(Ae^{i\tau} + zI) = g(A) \text{ for every } \tau \in \mathbb{R},\ z \in \mathbb{C}. \tag{4.7}$$

As it is proved in [8, Section 2.12],

$$\|(A - \lambda I)^{-1}\|_n \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A, \lambda)} \ \ (\lambda \notin \sigma(A)) \tag{4.8}$$

where $\sigma(A)$ means the spectrum of A, and

$$\rho(A, \lambda) = \min_{k=1,\dots,n} |\lambda - \lambda_k(A)|.$$

Hence we easily get

$$\Lambda_A \le \tilde{\Lambda}(A) := \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}|\alpha(A)|^{k+1}} \ \ (\lambda \notin \sigma(A))$$

where $\alpha(A) := \max_k \; Re\ \lambda_k(A)$. Moreover, thanks to Corollary 2.7.2 from [8]

$$\|e^{At}\|_n \leq p_A(t) \text{ where } p_A(t) := e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{g^k(A)}{(k!)^{3/2}} \quad (t \geq 0).$$

So

$$|e^{At}|_{L^2(R_+)} \leq [\int_0^\infty p_A^2(t)\, dt]^{1/2} \text{ and } |e^{At}|_{C(R_+)} \leq \max_{t\geq 0} p_A(t).$$

Clearly, the integral and maximum are simply calculated.

5. STABILITY OF EQUATIONS WITH DIFFERENTIAL LINEAR PARTS

In the following definition it is assumed that problem (4.1), (4.2) has at least one mild solution x for a given x_0.

Definition 5.1. Let $(F0)(t) \equiv 0$. Then the zero solution of (4.1) is said to be stable (in the Liapunov sense), if for any $\epsilon > 0$, there exists a $\delta > 0$, such that the condition $\|x_0\|_n \leq \delta$ implies the inequality $|x|_{C(R_+)} \leq \epsilon$ for any mild solution x of (4.1), (4.2).

The zero solution of (4.1) is said to be L^2-stable if it is stable, and there is an open set $B \subseteq \mathbb{C}^n$, such that $x_0 \in B$ implies $x \in L^2(R_+)$. Besides, B is called the region of attraction of the zero solution.

If the zero solution of (4.1) is L^2-stable and $B = \mathbb{C}^n$, then the zero solution is said to be globally L^2-stable.

Equation (4.1) is said to be absolutely L^2-stable in the class of nonlinearities (2.2) if under (2.2), there is a constants M which does not depend on a concrete form of F (but which depends on q) such that $|x|_{L^2(R_+)} \leq M\|x_0\|_n$ for any mild solution x of (4.1), (4.2).

From Theorem 4.1 it follows

Corollary 5.2. *Let conditions (3.1) and (4.4) hold. Then the zero solution of (4.1) is L^2-stable. Moreover, if condition (4.4) holds, then (4.1) is L^2-absolutely stable in the class of nonlinearities (2.2).*

Note that the above derived solution estimates give us bounds for the region of attraction.

Furthermore, consider the equation

$$\dot{x}(t) - Ax = F(x) + u(t). \tag{5.1}$$

where F maps $\Omega(r)$ into $L^2(R_+)$ and $u : R_+ \to \mathbb{C}$ is given. This equation under the condition

$$x(0) = 0 \tag{5.2}$$

is equivalent to (2.6) with

$$Q(t) = \exp\,[At] \text{ and } f(t) = \int_0^t e^{A(t-s)} u(s) ds. \tag{5.3}$$

In this case we also define a mild solution of (5.1), (5.2) as a solution of (2.6) with (5.3) taken into account. In the next definition the existence of mild solutions to (5.1), (5.2) is assumed.

Definition 5.3. Equation (5.1) is said to be input-to-state L^2-stable, if under (5.2), for any $\epsilon > 0$, there is a $\delta > 0$, such that the condition $|u|_{L^2(R_+)} \le \delta$ implies $|x|_{L^2(R_+)} \le \epsilon$.

Equation (5.1) is said to be globally input-to-state L^2-stable if the conditions (5.2) and $u \in L^2(R_+)$ imply that any mild solution is in $L^2(R_+)$.

Corollary 3.4 implies

Corollary 5.4. *If conditions (3.1) and (4.4) hold, then equation (5.1) is input-to-state L^2-stable.*

6. EQUATIONS WITH DIFFERENCE LINEAR PARTS

Consider the scalar difference equation

$$\sum_{k=0}^{m} C_k x(t-k) = [Fx](t) \ \ (t > 0) \tag{6.1}$$

with the continuous time t, constant $n \times n$-matrices C_k $(k = 1, \dots, m)$, $C_0 = I$, and the initial condition

$$x(t) = \phi(t) \ \ (-m \le t \le 0). \tag{6.2}$$

Here ϕ is a continuous function defined on $[-m, 0]$.

Put $Y(\lambda) = \lambda^m + C_1\lambda^{m-1} + \cdots + C_m$. It is assumed that all the zeros of $Y(\lambda)$ are inside the disc $|z| < 1$. Set

$$w(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(m+t)} Y^{-1}(e^{-i\omega})\, d\omega = \frac{1}{2\pi i} \int_{|z|=1} z^{m+t-1} Y^{-1}(z)\, dz.$$

By the Laplace transform one can easily rewrite (6.1) as (2.6) with $Q(t) = w(t)$ and $f(t) = y(t)$, where $y(t)$ is a solution of the homogeneous equation

$$\sum_{k=0}^{m} C_k y(t-k) = 0. \tag{6.3}$$

Denote

$$\theta_Y := \sup_{s \in \mathbb{R}} \|Y^{-1}(e^{-is})\| = \sup_{|z|=1} \|Y^{-1}(z)\|_n.$$

Clearly problem (6.1), (6.2) has a piece-wise continuous solution

Theorem 6.1. *Let the conditions (2.2) and $q\theta_Y < 1$ hold. Then any solution x of problem (6.1), (6.2) satisfies the inequality $|x|_{L^2(R_+)} \leq M_Y \ |\phi|_{C(-m,0)}$, where the constant M_Y does not depend on the initial conditions.*

Indeed, since (6.3) is stable, $|y|_{L^2(R_+)} \leq const \ |\phi|_{C(-m,0)}$. Now the result is due to Corollary 2.4. □

Note that by (4.8) one can derive bounds for θ_Y.

7. EQUATIONS WITH DIFFERENTIAL-DELAY LINEAR PARTS

Let $W(\tau)$ be a left-continuous $n \times n$-matrix-valued function defined on $[0, \eta]$ ($\eta < \infty$), whose entries are nondecreasing functions having bounded variations. Consider in $\mathbb{C}^n$ the equation

$$\dot{x}(t) = \int_0^\eta dW(\tau) x(t-\tau) + [Fx](t) \ \ (t > 0), \tag{7.1}$$

where F is a continuous causal mapping in $L^2(R_+)$.

Below we also consider nonlinear operators with delay acting from $L^2(-\eta, \infty)$ into $L^2(R_+)$.

Take the initial condition

$$x(t) = \phi(t) \ \ (-\eta \leq t \leq 0) \tag{7.2}$$

with a given continuous vector valued function ϕ defined on $[-\eta, 0]$. Introduce the characteristic matrix

$$K(z) = zI - \int_0^\eta e^{-z\tau} dW(\tau) \ (z \in \mathbb{C}).$$

A number λ is called *a characteristic value* of $K(\cdot)$ if $det\ K(\lambda) = 0$. Everywhere below it is assumed that *all the characteristic values of $K(\cdot)$ are in the open left half-plane.* That is, $K(\cdot)$ is stable.

Let

$$v(W) := \int_0^\eta \|dW(s)\|_n$$

be the variation of W. That is, $v(W)$ is the limit of the sums

$$\sum_{k=1}^n \|W(t_k) - W(t_{k-1})\|_n \ \ (0 \leq t_1 \leq \cdots \leq t_n = \eta)$$

as $\max_k |t_k - t_{k-1}| \to 0$. The following quantity plays a key role hereafter:

$$\theta(K) := \max_{-2v(W) \leq s \leq 2v(W)} \|K^{-1}(is)\|_n.$$

Consider the linear equation

$$\dot{y}(t) = \int_0^\eta W(d\tau) y(t-\tau). \tag{7.3}$$

The Green function (the fundamental solution) to equation (3.1) is a matrix-valued function $G(t)$ satisfying that equation and the initial conditions

$$G(0+) = I,\ G(t) = 0\ (t < 0).$$

We have

$$G(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ti\omega}K^{-1}(i\omega)d\omega\ (t \ge 0).$$

By the variation of constants formula problem (7.1), (7.2) is equivalent to the equation

$$x(t) = y(t) + \int_0^t G(t-t_1)(Fx)(t_1)dt_1\ \ (t > 0),$$

where y is a solution of (7.3).

A continuous solution of the latter integral equation will be called *a mild solution of problem (7.1), (7.2)*.

Theorem 7.1. *Let $F : \Omega(r) \to L^2(R_+)$ be a causal mapping continuous in the L^2-norm. Let the conditions (3.1) and*

$$q\theta(K) < 1 \tag{7.4}$$

hold. Then there are positive constants a_r and M_0, independent of the initial function and explicitly pointed below, such that the condition $a_r|\phi|_{C(-\eta,0)} < r$ provides the existence of a mild solution $x \in \Omega(r)$ of problem (7.1), (7.2), and

$$|x|_{L^2(R_+)} \le M_0|\phi|_{C(-\eta,0)},\ \text{and}\ |x|_{C(R_+)} \le a_r|\phi|_{C(-\eta,0)}.$$

To prove this theorem we need the following

Lemma 7.2. *The equality*

$$\sup_{s\in R^1}\|K^{-1}(is)\|_n = \sup_{|s|\le 2v(W)}\|K^{-1}(is)\|_n \equiv \theta(K)$$

is true.

Proof. We have $K(0) = W(\eta) - W(0)$. Since the entries of W are nondecreasing, this means that $\|K(0)\|_n = v(W)$. Hence

$$\|K^{-1}(0)\|_n = \sup_{w\in\mathbb{C}^n}\frac{\|K^{-1}(0)w\|_n}{\|w\|_n} =$$

$$\sup_{w_1\in\mathbb{C}^n}\frac{\|w_1\|_n}{\|K(0)w_1\|_n} \ge \frac{1}{\|K(0)\|_n} = 1/v(W).$$

But

$$\|K^{-1}(iy)\|_n \le 1/(|y| - v(W)) \le 1/v(W)\ \ (|y| \ge 2v(W)).$$

Thus the maximum of $\|K^{-1}(iy)\|_n$ is attained on $[-2v(W), 2v(W)]$. This proves the result. □

Proof of Theorem 7.1: The Parseval equality implies

$$|G|_{L^2(R_+)} = m_2(K) := [\frac{1}{2\pi}\int_{-\infty}^{\infty} \|[K(i\omega)]^{-1}\|_n^2 d\omega]^{1/2}.$$

For simplicity put $\theta(K) = \theta, m_2(K) = m_2$. Since (7.3) is stable, there are constants k_L and k_C, such that

$$|y|_{L^2(R_+)} \le k_L|\phi|_{C(-\eta,0)} \text{ and } |y|_{C(R_+)} \le k_C|\phi|_{C(-\eta,0)}. \tag{7.5}$$

Now thanks to to Corollary 3.4, we get

$$|x|_{L^2(R_+)} \le \frac{k_L|\phi|_{C(-\eta,0)}}{1-\theta(K)q} \text{ and}$$

$$|x|_{C(R_+)} \le |\phi|_{C(-\eta,0)}(k_C + \frac{k_L q|G|_{L^2(R_+)}}{1-\theta(K)q}). \tag{7.6}$$

So one can take

$$a_r = k_C + \frac{k_L q|G|_{L^2(R_+)}}{1-\theta(K)q} \text{ and } M_0 = \frac{k_L}{1-\theta(K)q}. \tag{7.7}$$

This proves the required result. □

Let us reduce to the form (2.6), the nonlinear differential delay equation

$$x(t) + \int_0^\eta dW(\tau)x(t-\tau) = \int_0^\eta d_s\nu(t,s)F_0(x(t-s)) \quad (t>0) \tag{7.8}$$

where ν is a matrix whose entries are nondecreasing in $s \in [0,\eta]$ functions of bounded variations and $F_0 : \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function. We can write down

$$\int_0^\eta d\nu(t,s)F_0(x(t-s)) = [Fx](t) + f_1(t)$$

where

$$[Fx](t) = \int_0^\eta d\nu(t,s)F_0(x(t-s)) = \int_{t-\eta}^t d\nu(t,t-\tau)F_0(x(\tau));\ f_1(t)\equiv 0 \quad (t>\eta)$$

and

$$[Fx](t) = \int_0^t d\nu(t,t-\tau)F_0(x(\tau));\ f_1(t) = \int_{t-\eta}^0 d\nu(t,t-\tau)F_0(\phi(\tau))\ (0\le t\le\eta).$$

So (7.8) can be written as (2.6) with $Q(t) = G(t)$ and

$$f(t) = y(t) + \int_0^t G(t-s)f_1(s)ds.$$

Thus the results of the present section are valid for equation (7.6).

8. ESTIMATES FOR GREEN'S FUNCTIONS OF DIFFERENTIAL-DELAY EQUATIONS

Recall that equation (7.3) is assumed to be exponentially stable and G is its Green function.

Lemma 8.1. *Let u be a solution of the equation*

$$\dot{u}(t) = \int_0^\eta dW(s)u(t-s) + f(t) \ \ (f \in L^2(R_+))$$

with the initial condition $u(t) = 0, t < 0$. Then

$$|\dot{u}|_{L^2(R_+)} \leq |u|_{L^2(R_+)} v(W) + |f|_{L^2(R_+)}.$$

Proof. Clearly,

$$|\dot{u}|_{L^2(R_+)} \leq |\int_0^\eta dW(s)u(t-s)|_{L^2(R_+)} + |f|_{L^2(R_+)}$$
$$= [\int_0^\infty \| \int_0^\eta dW(s)u(t-s)\|_n^2 dt]^{1/2} + |f|_{L^2(R_+)}.$$

But $u(t-s) = 0$ $(t < s)$. Hence,

$$|\dot{u}|_{L^2(R_+)} \leq (\int_0^\eta \|dW(s)\|_n) \sup_{-\eta \leq s \leq 0} [\int_0^\infty \|u(t-s)\|_n^2 dt]^{1/2} + |f|_{L^2(R_+)}$$
$$= v(W) \ |u|_{L^2(R_+)} + |f|_{L^2(R_+)}.$$

As claimed. □

Corollary 8.2. *The derivative of the Green function to equation (7.3) satisfies the inequality*

$$|\dot{G}|_{L^2(R_+)} \leq |G|_{L^2(R_+)} v(W).$$

Lemma 8.3. *The inequality $|G|_{L^2(R_+)} \leq \psi(K)$ is true, where*

$$\psi(K) := \sqrt{2\theta(K)(1 + v(W)\theta(K))}.$$

Proof. We can write down

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^\infty [K(i\omega)]^{-1} \exp(i\omega t)(i\omega + c)(i\omega + c)^{-1} \, d\omega.$$

with a positive constant c. Hence, $G(t) = \dot{w}(t) + cw(t)$, where

$$w(t) = \frac{1}{2\pi} \int_{-\infty}^\infty [K(i\omega)]^{-1} \exp(i\omega t)(i\omega + c)^{-1} d\omega.$$

Since e^{-ct} is the Laplace original for $(p+c)^{-1}$, it is not hard to check that $w(t)$ is a solution of the equation

$$\dot{w}(t) = \int_0^\eta dW(s)w(t-s) + Ie^{-ct}$$

with the zero initial condition $w(t) = 0$ for $t \le 0$. Since K is the Laplace transform of G, then by the Parseval equality we easily get the relations

$$|w|_{L^2(R_+)} \le \theta(K)|e^{-ct}|_{L^2(R_+)} = \theta(K)(2c)^{-1/2}.$$

Furthermore, thanks to the previous lemma and the latter inequality, one can write

$$|\dot{w}|_{L^2(R_+)} \le v(W)|w|_{L^2(R_+)} + |e^{-tc}|_{L^2(R_+)} \le \frac{v(W)\theta(K)+1}{\sqrt{2c}}.$$

Thus

$$|G|_{L^2(R_+)} \le |\dot{w}|_{L^2(R_+)} + c|w|_{L^2(R_+)} \le \frac{v(W)\theta(K)+1+c\theta(K)}{\sqrt{2c}}.$$

Taking $c = \theta^{-1}(K)(v(W)\theta(K)+1)$, we obtain the required inequality. □

From Lemma 8.3 now it follows

Corollary 8.4. *The derivative of the Green function to equation (7.3) satisfies the inequality*

$$|\dot{G}|_{L^2(R_+)} \le \psi(K)v(W).$$

To prove an estimate for the sup-norm of the Green function we will apply the following simple result: let a continuous vector-valued function h and its derivative $\dot{h}$ belong to space $L^2(R_+)$. Then

$$\|h(t)\|_n^2 \le 2[\int_t^\infty \|h(s)\|_n^2 ds \int_t^\infty \|\dot{h}(s)\|_n^2 ds]^{1/2} \ \ (t \ge 0).$$

For the proof see Lemma 8.7.4 from [7].

By this result and Corollaries 8.2 and 8.4 we get

Corollary 8.5. *The inequalities*

$$|G|_{C(R_+)} \le \sqrt{2v(W)}|G|_{L^2(R_+)} \le 2\sqrt{v(W)\theta(K)(1+v(W)\theta(K))}$$

are valid.

Denote

$$v_1(W) := \int_0^\eta s\|dW(s)\|_n.$$

Lemma 8.6. *Any solution y of (7.3) satisfies the inequalities*

$$|y|_{C(R_+)} \le |G|_{C(R_+)}(\|\phi(0)\|_n + v_1(W)|\phi|_{C(-\eta,0)}) \tag{8.1}$$

and

$$|y|_{L^2(R_+)} \le |G|_{L^2(R_+)}(\|\phi(0)\|_n + v_1(W)|\phi|_{C(-\eta,0)}). \tag{8.2}$$

Proof. Let us use the representation

$$y(t) = G(t)\phi(0) + \int_0^\eta dW(s) \int_{-s}^0 G(t-\tau-s)\phi(\tau)d\tau \tag{8.3}$$

cf. [12, 13], or [7, p. 148]. Hence,

$$|y|_{C(R_+)} \le |G|_{C(R_+)}(\|\phi(0)\|_n + |\phi|_{C(-\eta,0)} \int_0^\eta s\|dW(s)\|_n).$$

This proves (8.1). Similarly, (8.2) can be proved. □

According to the previous lemma inequalities (7.5) are true with

$$k_L \le \psi(K)(1+v_1(W)) \text{ and } k_C \le \psi(K)\sqrt{2v(W)}(1+v_1(W)). \tag{8.4}$$

This and (7.7) give us estimates for the region of attraction of the zero solution.

Denote, by $\Sigma(K)$ the spectrum of K. That is $\Sigma(K)$ is the set of the characteristic values of K. Thanks to (4.8),

$$\|K^{-1}(z)\|_n \le \Gamma(K(z)) \ \ (z \notin \Sigma(K)).$$

Here

$$\Gamma(K(z)) = \sum_{k=0}^{n-1} \frac{g^k(B(z))}{\sqrt{k!}d^{k+1}(K(z))} \ \ (z \notin \Sigma(K)),$$

where $d(K(z))$ is the smallest modulus of eigenvalues of $K(z)$:

$$d(K(z)) = \min_{k=1,\dots,n} |\lambda_k(K(z))|$$

for a fixed regular z; besides $\lambda_k(K(z))$ are the eigenvalues of matrix $K(z)$ counting with their multiplicities. We thus get.

Lemma 8.7. *Let equation (7.3) be stable. Then $\theta(K) \le \tilde{\theta}(K)$ where*

$$\tilde{\theta}(K) := \max_{|s|\le 2v(W)} \Gamma(K(is)).$$

From Lemma 8.3. we have

$$|G|_{L^2(R_+)} \le \tilde{\psi}(K) \text{ where } \tilde{\psi}(K) := \sqrt{2\tilde{\theta}(K)(1+v(W)\tilde{\theta}(K))}.$$

Moreover, Corollary 8.5 yields the inequality $|G|_{C(R_+)} \le \tilde{\psi}(K)\sqrt{2v(W)}$. Thus inequalities (7.5) are true with

$$k_L \le \tilde{\psi}(K)(1+v_1(W)) \text{ and } k_C \le \tilde{\psi}(K)\sqrt{2v(W)}(1+v_1(W)). \tag{8.5}$$

9. STABILITY OF EQUATIONS WITH DIFFERENTIAL-DELAY LINEAR PARTS

In this section we consider equations with causal nonlinearities acting in $L^2(R_+)$, but as it is shown in Section 7, nonlinearities acting from $L^2(+,\infty)$ into $L^2(R_+)$ can be easily reduced to the considered case.

Assume that problem (7.1), (7.2) has at least one mild solution x for a given ϕ. Let $(F0)(t) \equiv 0$. Following Definition 5.1 we will say that the zero solution of (7.1) is stable (in the Liapunov sense), if for any $\epsilon > 0$, there exists a $\delta > 0$, such that the condition $|\phi|_{C(-\eta,0)} \leq \delta$ implies the inequality $|x|_{C(R_+)} \leq \epsilon$ for any mild solution x of (7.1), (7.2).

The zero solution of (7.1) is said to be L^2-stable if it is stable, and there is an open set $B \subseteq \mathbb{C}^n$, such that $\phi \in B$ implies $x \in L^2(R_+)$. Besides, B is called the region of attraction of the zero solution.

If the zero solution of (7.1) is L^2-stable and $B = \mathbb{C}^n$, then the zero solution is said to be globally L^2-stable.

Equation (7.1) is said to be absolutely L^2-stable in the class of nonlinearities (2.2), if under (2.2) there is a constant M which does not depend on a concrete form of F (but which depends on q) such that $|x|_{L^2(R_+)} \leq M|\phi|_{C(-\eta,0)}$ for any mild solution x of (7.1), (7.2).

From Theorem 7.1 it follows

Corollary 9.1. *Let conditions (3.1) and (7.4) hold. Then the zero solution of (7.1) is L^2-stable. In addition, an initial function ϕ belongs to the region of attraction of the zero solution, provided*

$$a_r|\phi|_{C(-\eta,0)} < r,$$

where a_r is defined by (7.7).

Moreover, if condition (7.4) hold, then (7.1) is L^2-absolutely stable in the class of nonlinearities (2.2).

Recall that inequalities (8.4) for a_r are true.

Furthermore, consider the equation

$$\dot{x}(t) - \int_0^\eta dW(\tau)x(t-\tau) = F(x) + u(t), \tag{9.1}$$

where F maps $\Omega(r)$ into $L^2(R_+)$ and $u : R_+ \to \mathbb{C}^n$ is given. This equation under the condition

$$x(t) = 0 \ \ (t \leq 0) \tag{9.2}$$

is equivalent to (2.6) with

$$Q(t) = G(t) \text{ and } f(t) = \int_0^t G(t-s)u(s)ds. \tag{9.3}$$

In this case we also define a mild solution of (9.1), (9.2) as a continuous solution of (2.6) with (9.3) taken into account.

Following Definition 5.3, we will say that equation (9.1) is input-to-state L^2-stable, if under condition (9.2), for any $\epsilon > 0$, there is a $\delta > 0$, such that condition $|u|_{L^2(R_+)} \le \delta$ implies $|x|_{L^2(R_+)} \le \epsilon$. Equation (9.1) is said to be globally input-to-state L^2-stable if conditions (9.2) and $u \in L^2(R_+)$ imply that any mild solution of (9.1) is in $L^2(R_+)$.

Corollary 3.3 implies

Corollary 9.2. *If conditions (3.1) and (7.4) hold, then equation (9.1) is input-to-state L^2-stable.*

Moreover, if conditions (2.2) and (7.4) hold, then equation (9.1) is globally input-to-state L^2-stable.

10. LINEAR PARTS WITH ONE DELAY

In this section we illustrate our results in the case of equations with one delay in linear parts. First let us consider the linear equation

$$\dot{y}(t) = Ay(t-h) \ \ (t>0) \tag{10.1}$$

where A is a constant $n \times n$-matrix and $0 < h < \infty$. So in the considered case $\eta = h$ and the eigenvalues of K are $\lambda_j(K(z)) = z - e^{-zh}\lambda_j(A)$ since $K(z) = zI - e^{-zh}A$. In addition,

$$v(W) = \|A\|_n \text{ and } v_1(W) = h\|A\|_n.$$

According to (4.8), we have the inequality $\theta(K) \le \theta_A$, where

$$\theta_A := \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}d^{k+1}(K)},$$

and

$$d(K) := \inf_{j=1,\dots,n;\ |y|\le 2\|A\|} |yi + \lambda_j(A)e^{-iyh}|.$$

Now Lemma 8.3 and Corollary 8.5 imply

Lemma 10.1. *Let equation (10.1) be stable. Then its Green function is subject to the inequalities*

$$|G|_{L^2(R_+)} \le \psi_A := \sqrt{2\theta_A(1 + \|A\|_n\theta_A)} \ \textit{and}\ |G|_{C(R_+)} \le \psi_A\sqrt{2\|A\|_n}.$$

In particular, if A is a normal matrix, then $g(A) = 0$ and $\theta_A = 1/d(K)$. Let us use the following result.

Lemma 10.2. *Let $h, a_0 \in (0,\infty)$ and $a_0 h < \pi/4$. Then*

$$\inf_{\omega\in\mathbb{R}} |i\omega + a_0 e^{-ih\omega}| \geq a_0 \ \cos 2ha_0.$$

For the proof see [10]. From the latter lemma we get: let all the eigenvalues of A be real and negative, and

$$h|\lambda_j(A)| < \pi/4 \ \ (j = 1, \dots, n). \tag{10.2}$$

Then

$$\tilde{d}_A = \min_{j=1,\dots,n} |\lambda_j(A)| \ \cos(2h\lambda_j(A)), \tag{10.3}$$

Due to Lemma 10.1 we get the following result.

Corollary 10.3. *Let A be a Hermitian negative definite matrix and (10.2) hold. Then the Green function of equation (10.1) satisfies the estimates*

$$|G|_{L^2(R_+)} \leq \tilde{\psi}_A := \sqrt{\frac{2}{\tilde{d}_A}(1 + \frac{\|A\|_n}{\tilde{d}_A})}, \quad |G|_{C(R_+)} \leq \tilde{\psi}_A \sqrt{2\|A\|_n}.$$

Now one can apply Theorem 7.1 and its corollaries to the equation

$$\dot{x}(t) = Ax(t-h) + [Fx](t). \tag{10.4}$$

In particular, Corollary 9.1 implies

Corollary 10.4. *Let a causal operator F continuously map $L^2(R_+)$ into itself and A be a negative Hermitian matrix, such that the conditions (10.2) and*

$$q < |\lambda_j(A)| \ \cos\,(2h\lambda_j(A))\ (j = 1, \dots, n)$$

hold. Then equation (10.4) is absolutely L^2-stable in the class of nonlinearities (2.2).

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