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EXISTENCE RESULTS FOR ONE-DIMENSIONAL DIRICHLET ϕ-LAPLACIAN BVPS: A FIXED POINT APPROACH

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ABSTRACT. The aim of this paper is to present new existence results for ϕ-Laplacian Dirichlet boundary value problems set on bounded intervals of the real line. The fixed point theory approach and continuation methods are used throughout. Generalizations of some previous results regarding second-order differential equations are obtained.

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1. INTRODUCTION

In this paper, we are concerned with the existence of solutions to the homogenous Dirichlet boundary value problem:

$$\begin{cases} -\left(\phi(u')\right)'(x) & = & f(x,u(x)), \quad 0<x<1 \\ \ u(0)=u(1) & = & 0. \end{cases} \tag{1.1}$$

where $f\colon\ [0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function and $\phi\colon\ \mathbb{R}\longrightarrow\mathbb{R}$ is an increasing homeomorphism such that $\phi(0)=0$ and

$$\phi\left(|s|\right)\leq\left|\phi\left(s\right)\right|,\ \forall\, s\in\mathbb{R}. \tag{1.2}$$

Example 1.1. $\phi(s)=s^3-s^2+\frac{s}{2}\cdot$

Remark 1.2. (a) (1.2) is equivalent to:

$$\phi(s)\leq-\phi(-s),\ \forall\, s\in\mathbb{R}.$$

(b) When ϕ is odd, equality holds in (1.2).

(c) We deduce from Assumption (1.2) that if $\psi=\phi^{-1}$, then

$$\left|\psi\left(s\right)\right|\leq\psi\left(|s|\right)\ \text{ for all } s\in\mathbb{R}. \tag{1.3}$$

We consider in Section 4 the case f is continuous and also depends on the first derivative.

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The purpose of this paper is to prove some existence results for Problem (1.1) under suitable conditions on the nonlinear functions f and ϕ. Our approach is based on the application of fixed point theorems and nonlinear continuation methods of Leray and Schauder type. Some of the results extend previous ones regarding second-order boundary value problems. By a solution to Problem (1.1), we understand a function $u \in C^1([0,1];\mathbb{R})$ such that $\phi(u')$ is absolutely continuous and Problem (1.1) is satisfied almost everywhere.

The model case

$$\phi(s) = \phi_p(s) = \begin{cases} |s|^{p-2}s, & \text{for } s \neq 0 \\ 0, & \text{for } s = 0 \end{cases}$$

where $p > 1$ is a fixed real number, corresponding to the so-called one-dimensional $p-$Laplacian, has been widely investigated in the literature (see [1, 3, 8] and the references therein). The difficulty is that operator ϕ_p is linear only for $p = 2$. Problem (1.1) originates from partial differential equations of the $p-$Laplacian equation for which one seeks for radial solutions on annular domains of the euclidian space $\mathbb{R}^n$ (see [4, 5]):

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(\|x\|, u) &= 0, \quad r < \|x\| < R \quad (x \in \mathbb{R}^n) \\ u &= 0, \quad \text{for } \|x\| = r,\ R. \end{cases}$$

To deal with Problem (1.1), García-Huidobro *et al* [10] have introduced the notion of upper and lower $\sigma-$condition on ϕ, namely

$$\limsup_{s\to+\infty} \frac{\phi(\sigma s)}{s} < +\infty,\ \forall\, \sigma > 1$$

and

$$\limsup_{s\to+\infty} \frac{\phi(\sigma s)}{s} > 1,\ \forall\, \sigma > 1.$$

Making use of the time-mapping approach, they were interested in the spectral study of the nonlinear eigenvalue problem:

$$\begin{cases} -(\phi(u'))'(x) &= \lambda\phi(u(x)), \quad 0 < x < 1 \\ u(0) = u(1) &= 0. \end{cases} \tag{1.4}$$

Roughly speaking, the asymptotic behavior of the nonlinear function f is compared to the spectrum of Problem (1.4) and Problem (1.1) is then discussed.

In [11], the Dirichlet boundary value problem associated with the nonlinear $\phi-$Laplacian differential equation:

$$(\phi(u'))'(x) + g(u(x)) = q(x, u(x), u'(x)), \quad x \in (a,b)$$

is investigated. The assumption is that g grows faster than ϕ (super-linear case), namely $\lim\limits_{|y|\to+\infty} \frac{g(y)}{\phi(y)} = +\infty$. The time-mapping approach is again used. A multiplicity result is also obtained in [9] when the limits $\lim_{y\to 0}\frac{g(y)}{\phi(y)}$ and $\lim_{y\to+\infty}\frac{g(y)}{\phi(y)}$ lie in some resonance intervals.

When both ϕ and f are odd, a complete description of the set of solutions to the autonomous equation

$$(\mathcal{E})\ -(\phi(u'))'(x) = f(u(x)),\quad 0<x<1$$

subject to homogenous Dirichlet boundary conditions is provided in [1]. Some hypotheses on f and ϕ are assumed. The Rabinowitz global bifurcation theory and the quadrature method are employed. Note that the latter method can be used only for autonomous $\phi-$Laplacian differential equations. This method is based on the study of the qualitative properties of the time-mapping function defined, in case of the $p-$Laplacian, by

$$T_f(s) = \frac{2}{q^{\frac{1}{p}}}\int_0^s \frac{dt}{[F(s)-F(t)]^{\frac{1}{p}}}$$

where $F(s)=\int_0^s f(t)\,dt$ and $\frac{1}{p}+\frac{1}{q}=1$. This function gives the time between two zeros of a solution to Equation $(\mathcal{E})$.

The existence of infinitely many solutions to Problem (1.1) is studied by means of generalized polar coordinates in [16], introducing thereby a new technique to tackle $\phi-$Laplacian boundary value problems when $\phi(s)$ behaves as a power $s^\alpha\,(\alpha>0)$. Some restrictive conditions on

$$\liminf_{y\to\ell}\frac{f(x,y)}{\phi(y)}\quad\text{and}\quad\limsup_{y\to\ell}\frac{f(x,y)}{\phi(y)}$$

uniformly on compact subsets of $[0,1]$ are assumed to prove existence of solutions in case either $\ell=0$ or $\ell=+\infty$.

In [5], the case where ϕ is odd and satisfies a lower $\sigma-$ condition and f is superlinear is considered, and existence of solutions is discussed in terms of the positive eigenvalue λ of Problem (1.4). The upper and lower solution method and degree theory approach are used.

The latter methods are also applied in [3] to discuss the solvability of $p-$Laplacian Dirichlet boundary value problems:

$$\begin{cases} -(|u'|^{p-2}u')'(x) &= f(x,u(x)),\quad 0<x<T\\ u(0)=u(T) &= 0.\end{cases}$$

When further f depends on the first derivative, a priori bounds for the derivative is obtained in [13] assuming a priori bounds on the solution itself. This is essential in order to obtain solution for the boundary value problem.

A class of $\phi-$Laplacian boundary value problems associated with the equation

$$(\phi(u'))'(x)+k(x)\phi(u')(x)+f(x,u(x),u'(x))=0,\quad a<x<b$$

is considered in [4], where the function k is bounded. The growth condition on f reads as

$$|f(x, y, z)| \leq \alpha|g(y)| + \beta|g(z)| + \gamma,$$

where the positive real numbers α, β, γ satisfy some restrictive condition. Unfortunately, the obtained results do not encompass the classical case $k \equiv 0$.

More recently, Rynne [14] has studied $p-$Laplacian problems with Sturm-Liouville boundary conditions and jumping nonlinearities depending on the solution and its derivative. The notion of half-eigenvalues together with the Fučík spectral theory are developed.

The fixed point theory approach which seems to be well suited for a wide class of second-order ordinary differential equations is extended in this work to $\phi-$Laplacian problems. The paper is organized as follows. Some preliminaries are given in section 2 in order to write Problem (1.1) as a fixed point problem for a mapping, denoted T in Section 3. In the latter section, we present some existence results in case f does not depend on the first derivative; various growth conditions are assumed on the nonlinear term f including the sub-linear and super-linear cases and the sum of increasing and nondecreasing functions. Section 4 is devoted to the study of the existence of solutions in the more general framework of a derivative depending right-hand side term. New growth conditions are assumed. Through the transformation $v' = \phi(u')$, another fixed point formulation is given for a mapping denoted S. Sections 3 and 4 supply independent and complementary existence results. Finally, in section 5, some examples of applications illustrate the main results of this work and simple criteria of existence are derived. Connection with already known results on $p-$Laplacian problems is also provided. The notation $:=$ means throughout to be defined equal to.

2. PRELIMINARIES

We note by $C^1([0,1];\mathbb{R})$ the Banach space of all continuously differentiable functions from $[0,1]$ into $\mathbb{R}$ with norm

$$\|u\|_1 = \max\left(\|u\|_0, \|u'\|_0\right)$$

where $\|u\|_0 = \sup\left(|u(x)|,\ 0 \leq x \leq 1\right).$

Let $E = \{u \in C^1([0,1];\mathbb{R}),\ u(0) = u(1) = 0\} := C_0^1([0,1];\mathbb{R})$. Then, for any $u \in E$ and any $x \in (0,1)$, there exists some $\eta \in \mathbb{R}$, $0 < \eta < x$ such that $u(x) = xu'(\eta)$. Therefore, $|u(x)| \leq |u'(\eta)|$. Hence, $\|u'\|_0 = \max(\|u\|_0, \|u'\|_0)$ and so, equipped with the norm $\|u\|_E = \|u'\|_0$, E is a Banach space. Further recall that $L^1([0,1];\mathbb{R})$ is the Lebesgue space of real integrable functions on $[0,1]$. The norm in this space is denoted by

$$|u|_1 = \int_0^1 |u(t)|\,dt.$$

The following definition is classical:

Definition 2.1. $f\colon\ [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function if

(i) the map $x \longrightarrow f(x,y)$ is measurable for all $y \in \mathbb{R}$.

(ii) the map $y \longrightarrow f(x,y)$ is continuous for almost every $x \in [0,1]$.

If further there exists $h \in L^1([0,1];\mathbb{R})$ such that $|f(x,y)| \le h(x)$, for a.e. $x \in [0,1]$ and for all $y \in \mathbb{R}$, then f is said L^1-Carathéodory.

In order to transform Problem (1.1) into a fixed point problem, we need to develop the following auxiliary lemmas:

Lemma 2.2. *For any $h \in L^1([0,1];\mathbb{R})$, there exists a unique $C \in \mathbb{R}$ such that*

$$\int_0^1 \phi^{-1}\left(C - \int_0^s h(\tau)\,d\tau\right) ds = 0. \tag{2.1}$$

Proof. From the properties of the function ϕ, the mapping

$$H\colon\ C \hookrightarrow \int_0^1 \phi^{-1}\left(C - \int_0^s h(\tau)\,d\tau\right) ds$$

is increasing, continuous and satisfies $\lim\limits_{C\to\pm\infty} H(C) = \pm\infty$; then, it is also an homeomorphism on the real line. ☐

Lemma 2.3. [4] *Consider the boundary value problem*

$$\begin{cases} -(\phi(v'))'(x) &=& h(x), \quad 0 < x < 1 \\ v(0) = v(1) &=& 0 \end{cases} \tag{2.2}$$

where $h \in L^1([0,1];\mathbb{R})$. Then Problem (2.2) has a unique solution given by

$$v(x) = \int_0^x \phi^{-1}\left(C - \int_0^s h(\tau)\,d\tau\right) ds$$

where C is uniquely determined by the relation (2.1). Moreover $\phi(v'(0)) = C$ and

$$|C| < \int_0^1 |h(s)|\,ds. \tag{2.3}$$

By the Ascoli-Arzéla theorem, we can easily prove

Lemma 2.4. *The operator $A\colon\ L^1([0,1];\mathbb{R}) \longrightarrow C^1([0,1];\mathbb{R})$ defined by:*

$$Au(x) = \int_0^x \phi^{-1}\left(C - \int_0^s u(\tau)\,d\tau\right) ds$$

is completely continuous.

Next, consider the operator $T\colon C^1([0,1];\mathbb{R}) \longrightarrow C^1([0,1];\mathbb{R})$ defined by

$$Tu(x) = \int_0^x \phi^{-1}\left(C - \int_0^s f(\tau, u(\tau))\, d\tau\right) ds, \tag{2.4}$$

where C is the unique solution of the equation

$$\int_0^1 \phi^{-1}\left(C - \int_0^s f(\tau, u(\tau))\, d\tau\right) ds = 0.$$

Then

$$(Tu)'(x) = \phi^{-1}\left(C - \int_0^x f(s, u(s))\, ds\right). \tag{2.5}$$

In particular $\phi((Tu)'(0)) = C$ and it is clear that fixed points of T are solutions for the boundary value problem (1.1) and conversely. Finally, we have

Corollary 2.5. *If f is L^1-Carathéodory, then the operator T is completely continuous.*

Proof. The Nemyts'kĭi operator $N\colon C^1([0,1];\mathbb{R}) \longrightarrow L^1([0,1];\mathbb{R})$ defined by $Nv(x) = f(x, v(x))$ is continuous by Lebesgue dominated convergence theorem. The operator $T = AN\colon C^1([0,1];\mathbb{R}) \longrightarrow C^1([0,1];\mathbb{R})$ is the composition of N with the completely continuous mapping A introduced in Lemma 2.4; whence it is completely continuous. □

3. EXISTENCE RESULTS

In this section, we consider existence of solutions to Problem (1.1) under various growth conditions on the nonlinear functions f and ϕ. Hereafter $\mathbb{R}^+ = [0, +\infty)$ refers to the set of nonnegative real numbers.

3.1. Local growth conditions.

Theorem 3.1. *Let f be L^1-Carathéodory. If one of the following hypotheses is verified, then the boundary value problem (1.1) has at least one solution $u \in C^1([0,1];\mathbb{R})$: Either*

($\mathcal{H}1$) $|f(x,y)| \leq q(x)F(y)$, *for a.e. $x \in [0,1]$ and any $y \in \mathbb{R}$ where the functions $q \in L^1([0,1];\mathbb{R}^+)$ and $F \in C^0(\mathbb{R};\mathbb{R}^+)$ satisfy*

$$\exists\, r_0 > 0, \quad |q|_1 \max_{|y| \leq r_0} F(y) \leq \phi(r_0).$$

Or

($\mathcal{H}2$) $|f(x,y)| \leq G(x,|y|)$, *for a.e. $x \in [0,1]$ and any $y \in \mathbb{R}$ where the function $G\colon [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous nondecreasing with respect to the second argument and verifies*

$$\exists\, r_0 > 0, \quad \int_0^1 G(x, r_0)\, dx \leq \phi(r_0).$$

Proof. We appeal to Schauder's fixed point theorem which we recall for the reader's convenience (see [6] Thm 8.8, p. 60, [15] Thm 2.3.7, p. 15, [17] Thm 2.A, p. 57):

Theorem A. *Let E be a Banach space and $K \subset E$ a bounded, closed, convex subset of E. If $T\colon\ K \longrightarrow K$ is a completely continuous operator, then T has a fixed point in K.*

Consider the closed ball:

$$B = \{u \in E,\ \|u\|_E \leq r_0\}$$

where r_0 is as introduced in Assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$ and check that $T(B) \subset B$. Let $u \in B$ and $v = Tu$. Then v satisfies

$$\left\{\begin{array}{rcl} -\left(\phi(v')\right)'(x) & = & f(x, u(x)), \quad 0 < x < 1 \\ v(0) = v(1) & = & 0. \end{array}\right. \tag{3.1}$$

Let $\theta \in (0,1)$ be such that $v'(\theta) = 0$. Integrating the equation in (3.1), we get since $\phi(0) = 0$

$$\phi(v'(x)) = \int_x^\theta f(t, u(t))\,dt.$$

With (1.3), it follows that

$$\begin{array}{rcl} |v'(x)| & = & \left|\phi^{-1}\left(\int_x^\theta f(t, u(t))\,dt\right)\right| \\ & \leq & \phi^{-1}\left(\int_0^1 |f(t, u(t))|\,dt\right). \end{array} \tag{3.2}$$

(a) Assume $(\mathcal{H}1)$. For any $x \in [0,1]$, we have by (3.2)

$$\begin{array}{rcl} |v'(x)| & \leq & \phi^{-1}\left(\int_0^1 q(t)F(u(t))\,dt\right) \\ & \leq & \phi^{-1}\left(|q|_1 \max\limits_{|y| \leq r_0} F(y)\right) \\ & \leq & r_0. \end{array}$$

Passing to the supremum, we get

$$\|v\|_E = \|Tu\|_E \leq r_0.$$

Therefore, the ball B is invariant under the map T, ending the proof of our claim.

(b) In case $(\mathcal{H}2)$ is fulfilled, we derive analogously the estimates:

$$\begin{array}{rcl} \|Tu\|_E & \leq & \phi^{-1}\left(\int_0^1 |f(t, u(t))|\,dt\right) \\ & \leq & \phi^{-1}\left(\int_0^1 G(t, |u(t)|)\,dt\right) \\ & \leq & \phi^{-1}\left(\int_0^1 G(t, r_0)\,dt\right) \\ & \leq & r_0. \end{array}$$

Since T is completely continuous by Corollary 2.5, the proof of Theorem 3.1 follows from Theorem A. □

Remark 3.2. (a) It can be checked that the following assumption implies $(\mathcal{H}1)$:

$$|f(x,y)| \leq q(x)[F_1(y)+F_2(y)], \text{ for a.e. } x \in [0,1] \text{ and any } y \in \mathbb{R}$$

with some functions $q \in L^1([0,1];\mathbb{R}^+)$, $F_1 \in C^0(\mathbb{R};(0,+\infty))$, $F_2 \in C^0(\mathbb{R};\mathbb{R}^+)$ such that F_1 is nonincreasing, $\frac{F_2}{F_1}$ is nondecreasing and it holds that

$$\exists\, r_0 > 0 : \quad \frac{\phi(r_0)}{F_1(-r_0)\left(1+\frac{F_2(r_0)}{F_1(r_0)}\right)|q|_1} \geq 1.$$

In addition, this assumption encompasses the case F_1 is nonincreasing and F_2 nondecreasing.

(b) As will be explained through Application 2 in Section 5, Assumption $(\mathcal{H}1)$ covers the sub-linear and super-linear cases for the $p-$Laplacian problem. Moreover, the trivial solution may be ruled out by the additional hypothesis $f(x,0) \not\equiv 0$.

(c) In case the function G in Assumption $(\mathcal{H}2)$ is variable-separated, it is easily seen that $(\mathcal{H}2)$ implies $(\mathcal{H}1)$.

(d) It is easily seen that a sufficient condition for Assumption $(\mathcal{H}2)$ be satisfied is given in the following

Corollary 3.3. *Problem (1.1) has at least one solution whenever*

$$|f(x,y)| \leq q_1(x)\phi(|y|)+q_2(x) \;\; \textit{for a.e. } x \in [0,1] \textit{ and any } y \in \mathbb{R}$$

with some integrable functions q_1 and q_2 satisfying

$$0 < \int_0^1 q_1(s)\,ds < 1 \;\; \textit{and} \;\; \int_0^1 q_2(s)\,ds \geq 0.$$

Corollary 3.4. *The boundary value problem:*

$$\begin{cases} -\left(\phi(u')\right)'(x) &= \lambda a(x)F(u(x)), \quad 0<x<1 \\ u(0)=u(1) &= 0 \end{cases}$$

where $a \in L^1([0,1];\mathbb{R})$, $F \in C^0(\mathbb{R};\mathbb{R})$ and

$$1 \leq \limsup_{s\to+\infty} \frac{-\phi(-s)}{\phi(s)} = \ell_\phi < \infty$$

$$0 < \limsup_{s\to+\infty} \frac{F(s)}{\phi(s)} = \ell_1 < \infty, \quad 0 < \limsup_{s\to-\infty} \frac{F(s)}{\phi(s)} = \ell_2 < \infty$$

admits a solution whenever $0 \leq \lambda < 1/\bar{\ell}\ell_\phi|a|_1$ with $\bar{\ell} := \max(\ell_1,\ell_2)$.

Proof. By assumption, there exists some positive constant M such that

$$|F(y)| \leq \bar{\ell}|\phi(y)|+M, \quad \forall\, y \in \mathbb{R}.$$

We know from Assumption (1.2) that for any positive number r_0, we have that $\max\limits_{[-r_0,r_0]} \bar{\ell}|\phi(y)|+M = \bar{\ell}|\phi(-r_0)|+M$. Assumption ($\mathcal{H}1$) in Theorem 3.1 is then fulfilled if there exists some r_0 such that

$$\lambda|a|_1 \left[\bar{\ell}|\phi(-r_0)|+M\right] \leq \phi(r_0)$$

that is

$$\lambda|a|_1\bar{\ell}\left(\frac{-\phi(-r_0)}{\phi(r_0)}\right)+M\frac{\lambda|a|_1}{\phi(r_0)} \leq 1.$$

This may always be achieved by taking $0 \leq \lambda < 1/\bar{\ell}\ell_\phi|a|_1$ and choosing r_0 sufficiently large. □

4. THE DERIVATIVE DEPENDING GENERAL CASE

In this second part of the paper, we provide existence of solutions for the following general boundary value problem:

$$\left\{\begin{array}{rcl} -\left(\phi(u')\right)'(x) &=& f(x,u,u'), \quad 0<x<1 \\ u(0)=u(1) &=& 0, \end{array}\right. \tag{4.1}$$

where $f\colon [0,1]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function. Fairly weak restrictions on both f and ϕ will be assumed in the sequel. In what follows, assume ϕ is an increasing homeomorphism from I onto $\mathbb{R}$. I is an open interval of $\mathbb{R}$ containing 0 and $\phi(0)=0$. The main two results in this section are:

Theorem 4.1. *Assume the interval I is bounded, then Problem* (4.1) *admits at least one solution.*

Note that surprisingly no growth assumption is made on the nonlinearity f. The reason is that the boundedness of I implies straightforward estimates of any solution. Moreover, ϕ need not satisfy Assumption (1.2).

The next result is an extension of the one obtained in [2] for second-order Dirichlet boundary value problems. By contrast to Theorem 4.1, Assumption (1.2) is now assumed to hold in order to prove the following

Theorem 4.2. *Assume $I=\mathbb{R}$ and the functions ϕ, f satisfy the following conditions:*

$$\left\{\begin{array}{l} \textit{There exists a continuous function } G\colon \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+ \\ \textit{increasing with respect to its variables such that} \\ |f(x,y,z)| \leq G(|y|,|z|) \textit{ for all } x\in[0,1] \textit{ and } y,z\in\mathbb{R}. \end{array}\right. \tag{4.2}$$

$$\left\{\begin{array}{l} \textit{There exists a real number } \alpha\geq 0 \textit{ such that the set} \\ \mathcal{A}_\alpha = \{s>0\colon\ \phi(s) \leq (\alpha+1)\,G(s,s)\} \neq \emptyset \textit{ and } \sup\mathcal{A}_\alpha<\infty. \end{array}\right. \tag{4.3}$$

Then Problem (4.1) *admits at least one solution $u\in C^1([0,1];\mathbb{R})$.*

Remark 4.3. (a) By replacing the dominating function G by $G+1$ if necessary, one may assume $G(0,0)>0$; by continuity, there exists some $s_0>0$ such that $G(s_0,s_0)>0$. Then for every $\alpha \geq \max\left(0,-1+\frac{\phi(s_0)}{G(s_0,s_0)}\right)$, the set $\mathcal{A}_\alpha$ is nonempty, showing that the first part in Condition (4.3) is always satisfied.

(b) A sufficient condition for the second part in (4.3) be satisfied is

$$\limsup_{s\to+\infty}\frac{(\alpha+1)G(s,s)}{\phi(s)}<1.$$

(c) The second part in condition (4.3) implies that

$$G(s,s)<\frac{\phi(s)}{\alpha+1}\leq\phi(s),\quad \forall\, s>\sup\mathcal{A}_\alpha,$$

showing that along the diagonal $s=t$, the function $G(s,t)$ should have sub-linear growth with respect to the $\phi-$Laplacian.

4.1. **Auxiliary lemmas.** Denote by X the Banach space consisting of all functions $u\in C^1([0,1];\mathbb{R})$ with $u(0)=0$ and endowed with norm

$$\|u\|_X=\|u'\|_0=\sup\{|u'(x)|:\ x\in[0,1]\}.$$

It is well known that the operator $L\colon C_0^2([0,1];\mathbb{R})\to C([0,1];\mathbb{R})$ defined by $Lu=-u''$ is invertible with inverse L^{-1} given by

$$L^{-1}v(x)=\int_0^1\mathcal{G}(x,t)v(t)\,dt$$

where $C_0^2([0,1];\mathbb{R})=\{u\in C^2([0,1];\mathbb{R}):\ u(0)=u(1)=0\}$ and $\mathcal{G}$ is the Green function given by

$$\mathcal{G}(x,t)=\begin{cases} x(1-t) & \text{for } 0\leq x\leq t\leq 1\\ t(1-x) & \text{for } 0\leq t\leq x\leq 1.\end{cases}\tag{4.4}$$

In fact, the substitution $v'=\phi(u')$ transforms the equation in Problem (4.1) into a strongly nonlinear second-order differential equation for the new unknown v. The proofs of Theorems 4.1 and 4.2 are based on the following classical result known as Leray-Schauder's continuation principle or Schäfer's Theorem (see [6] Corollary 8.1, p. 61 or [15] Thm 4.3.2, p. 29):

Theorem B. *Let X be a Banach space and let $S\colon X\to X$ be a completely continuous operator. Assume that there exists a real constant $R>0$ such that for all $u\in X,\lambda\in[0,1]$ and $u=\lambda Su$ implies $\|u\|<R$. Then S admits a fixed point in X.*

The function ψ being the inverse of ϕ, consider the operators A_ψ, B_ψ defined on X by

$$\begin{cases} A_\psi v(x) &= \int_0^x\psi(v'(s))\,ds\\ B_\psi v(x) &= \psi(v'(x))=(A_\psi v)'(x)\end{cases}\tag{4.5}$$

and next $S : X \to X$ be such that

$$Sv\,(x) = \int_0^1 \mathcal{G}(x,t) f(t, A_\psi v(t), B_\psi v(t))\, dt + x \int_0^1 (v'(t) - B_\psi v(t))\, dt. \tag{4.6}$$

Remark 4.4. It is easy to check that A_ψ maps X into itself, B_ψ maps X into $C\,([0,1]\,;\mathbb{R})$ and then S maps X into itself.

In a series of lemmas, we investigate the properties of these mappings.

Lemma 4.5. *Operators A_ψ and B_ψ defined above are continuous.*

Proof. It is clear that the claim of the lemma follows immediately if one shows that the operator B_ψ is continuous. Let $(v_n)_{n\geq 1}$ be a sequence in X converging to some $v \in X$ and let $M > 0$ be such that $(v_n)_{n\geq 1} \subset \overline{B}\,(0, M)$. Since ψ is continuous, it is uniformly continuous on the compact interval $[-M, M]$. Hence, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s_1,\, s_2 \in [-M, M]$, $|s_1 - s_2| < \delta$ implies $|\psi\,(s_1) - \psi\,(s_2)| < \varepsilon$. In addition, there exists some $n_0 \in \mathbb{N}$ such that

$$\forall\, n \in \mathbb{N},\ (n > n_0 \Rightarrow |v_n'\,(x) - v'\,(x)| < \delta),\ \forall\, x \in [0,1]$$

and then $|\psi\,(v_n'\,(x)) - \psi\,(v'\,(x))| < \varepsilon$, for any $x \in [0,1]$. It follows that

$$\|\psi\,(v_n') - \psi\,(v')\|_0 < \varepsilon,$$

proving the lemma. □

Lemma 4.6. *The operator S is completely continuous.*

Proof. Denote $S_1, S_2\colon\ X \to X$ the operators defined by

$$\begin{cases} S_1 v\,(x) &= x\int_0^1 (v'(t) - B_\psi v(t))\, dt \\ S_2 v\,(x) &= \int_0^1 \mathcal{G}(x,t) f(t, A_\psi v(t), B_\psi v(t))\, dt \end{cases} \tag{4.7}$$

for all $v \in X$ so that $S = S_1 + S_2$. Since S_1 has range lying in the $1-$dimensional vector space $[\{\mathrm{Id}_{|[0,1]}\}]$, it is obviously completely continuous. Then, it is enough to prove that the operator S_2 is completely continuous. For this, consider the diagram:

$$\begin{array}{ccccc} X & \xrightarrow{A_\psi}\longrightarrow & X & \xrightarrow{N} & C^0([0,1];\mathbb{R}) \\ \downarrow & & & & \downarrow \\ S_2 \downarrow & & & & \downarrow L^{-1} \\ \downarrow & & & & \downarrow \\ X & & \longleftarrow\overset{j}{\longleftarrow}\longleftarrow & & C_0^2([0,1]) \end{array} \tag{4.8}$$

where j is the compact embedding of $C_0^2\,([0,1]\,;\mathbb{R})$ into X, N is the Nemyts'kĭi operator defined for $x \in [0,1]$ and $v \in X$ by $Nv(x) = f(x, v(x), v'(x))$. Hence, the operator $S_2 = j \circ L^{-1} \circ N \circ A_\psi$ defined in (4.7) is completely continuous, proving our claim. □

Lemma 4.7. *If $v \in X$ is a fixed point of S, then the function u defined by $u(x) = A_\psi v(x) = \int_0^x \psi(v'(t))\,dt$ for all $x \in [0,1]$ is a solution to Problem* (4.1).

Proof. First, $u(0) = 0$. Putting $x = 1$ in (4.6), we get, since $\mathcal{G}(1,t) \equiv 0$, $t \in [0,1]$

$$Sv(1) = v(1) = \int_0^1 (v'(t) - \psi(v'(t)))\,dt = v(1) - \int_0^1 \psi(v'(t))\,dt$$

and so

$$u(1) = \int_0^1 \psi(v'(t))\,dt = 0.$$

Differentiating twice with respect to x the relation

$$v(x) = \int_0^1 \mathcal{G}(x,t) f(t, A_\psi v(t), B_\psi v(t))\,dt + x\int_0^1 (v'(t) - \psi(v'(t)))\,dt,$$

we get

$$(\phi(u'(x)))' = v''(x) = -f(x, A_\psi v(x), B_\psi v(x)) = -f(x, u(x), u'(x)),$$

proving the lemma. □

Lemma 4.8. *Let $v \in X$ be such that $v = \lambda S v$ for some $\lambda \in [0,1]$. Then*
(a) *v is solution of the problem with nonlocal right boundary condition*

$$\begin{cases} -v''(x) = \lambda f(x, A_\psi v(x), B_\psi v(x)), & x \in (0,1) \\ v(0) = 0 \text{ and } (\lambda - 1)v(1) = \lambda \int_0^1 \psi(v'(t))\,dt. \end{cases} \tag{4.9}$$

(b) *There exists $\tau \in [0,1]$ such that $v'(\tau) = 0$.*

Proof. **(a)** The function v satisfies the integral equation:

$$v(x) = \lambda \int_0^1 \mathcal{G}(x,t) f(t, A_\psi v(t), B_\psi v(t))\,dt + \lambda x \int_0^1 (v'(t) - \psi(v'(t)))\,dt. \tag{4.10}$$

Since $\mathcal{G}(0,.) = \mathcal{G}(1,.) \equiv 0$, we infer from (4.10) that $v(0) = 0$ and

$$(\lambda - 1)v(1) = \lambda \int_0^1 \psi(v'(t))\,dt.$$

Differentiating twice with respect to x the relation (4.10), we find

$$v''(x) = -\lambda f(x, A_\psi v(x), B_\psi v(x))$$

and thus v satisfies Problem (4.9), proving the first part.

(b) We claim that there exists some $\tau \in [0,1]$ such that $v'(\tau) = 0$. Indeed, if $v'(x) > 0$ for all $x \in [0,1]$ (the case $v'(x) < 0$ for all $x \in [0,1]$ is treated similarly), then in one hand we have $v(1) > v(0) = 0$ and in the other one the relation $(1 - \lambda)v(1) = -\lambda \int_0^1 \psi(v'(t))\,dt$ implies $v(1) < 0$, leading to a contradiction. □

Now, we are in position to prove Theorems 4.1 and 4.2. Let v and τ be as introduced in Lemma 4.8.

4.2. **Proof of Theorem 4.1.** Integrating (4.9) between τ and x, we find

$$\begin{aligned} |v'(x)| &= \lambda\left|\int_x^\tau f(t, A_\psi v(t), B_\psi v(t))\,dt\right| \\ &\leq \int_0^1 |f(x, A_\psi v(x), B_\psi v(x))|\,dx \\ &\leq \sup\left\{|f(x,y,z)| : (x,y,z) \in [0,1]\times[r_-,r_+]^2\right\} < \infty \end{aligned}$$

where $r_- : = \inf I$ and $r_+ : = \sup I$. Thus

$$\|v\|_X \leq \sup\left\{|f(x,y,z)| : (x,y,z) \in [0,1]\times[r_-,r_+]^2\right\}$$

and we deduce from Theorem B and Lemma 4.7 that the operator S admits a fixed point $v \in X$. Therefore $u = A_\psi v$ is a solution to Problem (4.1), ending the proof.

4.3. **Proof of Theorem 4.2.** The positive real number α being as introduced in Assumption (4.3), multiply the equation in (4.9) by $(v')^\alpha$ and integrate between τ and x; we get

$$\begin{aligned} \frac{1}{\alpha+1}|v'(x)|^{\alpha+1} &= \lambda\left|\int_x^\tau f(t, A_\psi v(t), B_\psi v(t))\,(v'(t))^\alpha\,dt\right| \\ &\leq \int_0^1 |f(x, A_\psi v(x), B_\psi v(x))|\,|v'(x)|^\alpha\,dx. \end{aligned}$$

Using (4.2) and Remark 1.2(c), we find that $|A_\psi v|$, $|B_\psi v| \leq |\psi(v')| \leq \psi(|v'|) \leq \psi(\|v\|)$ and thus

$$\begin{aligned} \frac{1}{\alpha+1}|v'(x)|^{\alpha+1} &\leq \int_0^1 G(|A_\psi v(x)|, |B_\psi v(x)|)\,|v'(x)|^\alpha\,dx \\ &\leq G(\psi(\|v\|_X), \psi(\|v\|)_X)\,\|v\|_X^\alpha. \end{aligned}$$

Passing to the supremum over $x \in [0,1]$, we get

$$\|v\|_X \leq (\alpha+1)\,G\left(\psi(\|v\|_X), \psi(\|v\|_X)\right).$$

The constant $s_0 = \psi(\|v\|_X)$ then satisfies the inequality:

$$\phi(s_0) \leq (\alpha+1)\,G(s_0, s_0).$$

From Assumption (4.3), we infer that $s_0 \leq R_0 : = \sup \mathcal{A}_\alpha$ and so $\|v\|_X \leq \phi(R_0)$. With Theorem B and Lemma 4.7, the operator S then admits at least one fixed point $v \in X$ and so $u = A_\psi v$ is a solution to Problem (4.1), ending the proof of the theorem.

Corollary 4.9. *Let a, h be continuous functions. Then, the problem:*

$$\begin{cases} -(\phi(u'))'(x) = \lambda a(x)\phi(u) + h, & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

has a solution for any $\lambda \in [0, \frac{1}{\ell_\phi \|a\|_0})$ where $1 \leq \ell_\phi = \limsup\limits_{s\to+\infty} \frac{-\phi(-s)}{\phi(s)} < \infty$.

Proof. Define the function

$$\tilde{\phi}(s) = \begin{cases} -\phi(s), & \text{for } s \leq 0 \\ -\phi(-s), & \text{for } s \geq 0. \end{cases}$$

From Assumption (1.2), we know that $\phi(s) \leq \tilde{\phi}(s)\ \ \forall s \in \mathbb{R}$. Moreover $\tilde{\phi}$ is positive, even and satisfies $\tilde{\phi}(s) = -\phi(-|s|)$, $|\tilde{\phi}(s)| = \tilde{\phi}(s)$ and $|\phi(s)| \leq \tilde{\phi}(s)$.

Let $G(s) \colon = \lambda a_0 \tilde{\phi}(s) + h_0$ where $a_0 = \|a\|_0$ and $h_0 \colon = \|h\|_0$. Then, Assumption (b) in Remark 4.3 is fulfilled with $\alpha = 0$ whenever $0 \leq \lambda < \frac{1}{\ell_\phi a_0}\cdot$ Indeed

$$\lim_{s\to+\infty} \frac{\lambda a_0 \tilde{\phi}(s) + h_0}{\phi(s)} = \lim_{s\to+\infty} \frac{-\lambda a_0 \phi(-s) + h_0}{\phi(s)} = \lambda a_0 \ell_\phi.$$

□

Remark 4.10. When $h = 0$, Corollary 4.9 is not really a Fredholm-like result for the obtained solution may be zero. For instance, in case ϕ is the $p-$Laplacian, the spectral problem with $a \equiv 1$ only yields the trivial solution. Indeed, we know [8] that the eigenvalues for the $p-$Laplacian are $\lambda_n = (p-1)(n\pi_p)^p,\ n \in \mathbb{N}^*$ with $\pi_p = \frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}$; thus for any $p \geq \frac{\ln 3}{\ln 2}$, we have that $\lambda_n > 1,\ \forall n \in \mathbb{N}^*$. For the $p-$Laplacian, the spectral study of the problem associated with the equation $(\mathcal{E}) : -(\phi_p(u'))' = \lambda\phi_p(u)$ is well investigated in the literature. A Fredholm-like result for the inhomogeneous problem associated with Equation $(\mathcal{E})$ is given in [7, 11]. A description of what is called pseudo-eigenvalues of Problem $(\mathcal{E})$ has been also discussed in [10] when the nonlinear function ϕ is asymptotic to a power.

5. **APPLICATIONS**

5.1. **Application 1: $\phi-$Laplacian BVPs.** Consider the ϕ-Laplacian boundary value problem

$$\begin{cases} -(\phi(u'))'(x) &=& a(x) + b(x)\phi(|u|), \quad 0 < x < 1 \\ u(0) = u(1) &=& 0 \end{cases} \tag{5.1}$$

where $a,\, b \in L^1([0,1];\mathbb{R})$ and $\int_0^1 |b(x)|\, dx < 1$. We have that $|a(x) + b(x)\phi(|u|)| \leq G(x,|u|) = |a(x)| + |b(x)|\phi(|u|)$. The function $G(x,s) \colon = |a(x)| + |b(x)|\phi(s)$ is nondecreasing with respect to the second argument and satisfies for every positive r:

$$\int_0^1 G(x,r)\, dx = |a|_1 + |b|_1\phi(r) \leq \phi(r) \Leftrightarrow r \geq \phi^{-1}\left(\frac{|a|_1}{1-|b|_1}\right).$$

Assumption $(\mathcal{H}2)$ of Theorem 3.1 is then fulfilled yielding existence of a nontrivial solution to Problem (5.1) whenever $a \not\equiv 0$. However, $|a(x)+b(x)\phi(|u|)| \leq c(x)\,(1+\phi(|u|))$ with $c(x) \colon = \max(|a(x), |b(x)|)$. If $\int_0^1 |a(x)|\, dx \geq 1$, then the inequality in Assumption $(\mathcal{H}1)$ is never satisfied for

$$|c|_1\,(1+\phi(r)) \geq 1 + \phi(r) > \phi(r), \quad \forall r > 0.$$

This shows the independence and complementarity of Assumptions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ of Theorem 3.1 in the non-autonomous case.

5.2. **Application 2: $p-$Laplacian BVPs.** Assume $\phi(s) = |s|^{p-2}s$ $(p > 1,\ s \neq 0)$ and $|f(x,y)| \leq q(x)(1+|y|^{\sigma})$ for a.e. $x \in [0,1]$ and any $y \in \mathbb{R}$ where $q \in L^1([0,1];\mathbb{R}^+)$ and σ is a positive real number. Then, the inequality in Assumption $(\mathcal{H}1)$ of Theorem 3.1, namely:

$$\text{there exists } r_0 > 0 \text{ such that } |q|_1 \leq r_0^{p-1-\sigma}$$

is satisfied for any σ and p such that either $0 < \sigma \leq p-1$ or $\sigma > p-1$ and

$$|q|_1 < \frac{\sigma - p + 1}{\sigma}\left(\frac{p-1}{\sigma - p + 1}\right)^{\frac{p-1}{\sigma}} = (p-1)\left(\frac{p-1}{\sigma - p + 1}\right)^{\frac{p-1-\sigma}{\sigma}}.$$

This proves existence of solutions in the sub-linear as well as in the super-linear growth case. Of course, such a solution is nontrivial whenever equality $|f(x,y)| = q(x)(1+|y|^{\sigma})$ holds true.

5.3. **Application 3: Second-order BVPs.** Consider the Dirichlet boundary value problem for the class of second-order differential equations

$$\begin{cases} -u''(x) = h(x,u(x))g(u'(x)), & 0 < x < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{5.2}$$

where $h\colon\ [0,1]\times\mathbb{R} \to \mathbb{R}$ and $g\colon\ \mathbb{R} \to [0,+\infty)$ are continuous functions. Let ϕ be formally defined by

$$\phi(s) = \int_0^s \frac{dt}{g(t)}. \tag{5.3}$$

It is clear that u is a solution to Problem (5.2) if u is a solution to

$$\begin{cases} -\left(\phi(u')\right)'(x) = h(x,u(x)), & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{5.4}$$

and such a solution is nonnegative and concave if h only takes positive values. Hereafter, we prove two distinct existence results for Problem (5.2) depending on whether or not the function g vanishes on the real line.

Corollary 5.1. *Assume there exist $r_- < 0 < r_+$ such that $g(r_-) = g(r_+) = 0$, $g > 0$ in (r_-, r_+) and*

$$\lim_{s\to r_\pm} \int_0^s \frac{dt}{g(t)} = \pm\infty. \tag{5.5}$$

Then Problem (5.2) admits at least one solution $u \in C^1([0,1];\mathbb{R})$.

Proof. The function ϕ given by (5.3) is well defined and is an homeomorphism from (r_-, r_+) into $\mathbb{R}$. A direct application of Theorem 4.1 yields a solution to Problem (5.4) ending the proof of the corollary. ∎

Remark 5.2. As noticed in Theorem 4.1, Assumption (1.2) is not needed here. However, if g is even, then ϕ is odd.

Remark 5.3. (a) In [12], it is proved that the problem

$$\begin{cases} u''(x) &= g(u'(x)), \quad 0 < x < 1 \\ u(0) &= u(1) = 0. \end{cases}$$

has a solution if g has two zeros of opposite signs. This result is also recovered in [2] without Condition (5.5). Corollary 5.1 is then an extension of this result for h may be any continuous function.

(b) The positivity of g may be relaxed; indeed if g has many zeros, it takes constant sign between two of them.

(c) The trivial solution may be ruled out by imposing the additional condition $h(x,0) \not\equiv 0$.

Corollary 5.4. *The function ϕ being as defined by (5.3), assume the functions g and h fulfill the following conditions:*

$$g(s) > 0 \text{ and } g(s) \le g(|s|) \text{ for all } s \in \mathbb{R}. \tag{5.6}$$

$$\int_0^{+\infty} \frac{dt}{g(t)} = +\infty. \tag{5.7}$$

$$\limsup_{|s|\to+\infty} \frac{|h(x,s)|}{\phi(|s|)} < 1 \text{ uniformly in } x \in [0,1]. \tag{5.8}$$

Then Problem (5.2) admits at least one solution $u \in C^1([0,1];\mathbb{R})$.

Proof. We will prove that the functions ϕ and h satisfy the conditions of Theorem 4.2. We have that $s\phi(s) \ge 0$ for all $s \in \mathbb{R}$. Then, for $s < 0$ we obtain after easy computations:

$$\begin{aligned} \phi(|s|) - |\phi(s)| &= \int_0^{-s} \frac{dt}{g(t)} + \int_0^{s} \frac{dt}{g(t)} \\ &= \int_s^0 \frac{dt}{g(-t)} - \int_s^0 \frac{dt}{g(t)} \\ &= \int_s^0 \frac{g(t)-g(-t)}{g(t)g(-t)}\,dt \le 0 \end{aligned}$$

and thus Condition (1.2) holds true. Finally, from Assumption (5.8), we infer that there exist some $\gamma \in (0,1)$ and $\delta > 0$ such that

$$|h(x,y)| \le \gamma\phi(|y|) + \delta \text{ for all } (x,y) \in [0,1] \times \mathbb{R}.$$

Let $G(s) := \gamma\phi(s) + \delta$. Then the set $\mathcal{A}_\alpha = \{s > 0\colon \phi(s) \le G(s)\}$, which coincides with the interval $\left(0, \psi\left(\frac{\delta}{1-\gamma}\right)\right]$, is not empty and bounded; thus Assumption (4.3) is satisfied, ending thereby the proof of the corollary. □

Remark 5.5. Assumption (5.7) ensures that ϕ is an homeomorphism on the real line. In particular, $\phi(+\infty) = +\infty$ while $\phi(-\infty) = -\infty$ follows from $\phi(s) \leq -\phi(-s)$ valid for any $s < 0$. Although Assumption (5.7) is the usual strong Nagumo-Bernstein condition, we need not impose existence of upper and lower solution to prove existence of solution as the following example illustrates where the nonlinearity $f(x, y, z)$ is only required to have sub-linear growth with respect to the product yz:

Example 5.6. The boundary value problem

$$\begin{cases} -u''(x) &= a(x)(1 + |u(x)|^\beta)(1 + |u'(x)|^\alpha), \quad 0 < x < 1 \\ u(0) &= u(1) = 0 \end{cases}$$

with $a \in C^0([0, 1]; \mathbb{R}^+)$, $a \not\equiv 0$, has a nontrivial positive solution provided

$$\text{either } 0 < \alpha + \beta < 1$$

$$\text{or } \ 0 < \alpha + \beta = 1 \text{ and } 0 < \max_{0 \leq x \leq 1} a(x) < 1.$$

Indeed,

$$\begin{aligned} \tfrac{h(x,s)}{\phi(|s|)} &= a(x) \frac{1 + |s|^\beta}{\int_0^{|s|} \frac{dt}{1+|t|^\alpha}} \\ &\leq a(x) \frac{(1 + |s|^\beta)(1 + |s|^\alpha)}{|s|} \sim a(x)|s|^{\alpha+\beta-1}, \quad \text{as } |s| \to +\infty. \end{aligned}$$

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