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OSCILLATION FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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ABSTRACT. Some new oscillation criteria are given for second order nonlinear differential equations with damping of the form $(r(t)x')' + p(t)x' + q(t)f(x) = 0$. Our results are to develop oscillation criteria without any restriction on the signs of $p(t)$ and $q(t)$. These results generalize and extend some earlier results of Abdullah [1] and Zheng and Liu [12].

Keywords and Phrases. Differential equations; second order; nonlinear; damping; oscillation

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1. INTRODUCTION

In this paper, we consider oscillatory properties for the second order nonlinear differential equation with damped term

$$(r(t)x')' + p(t)x' + q(t)f(x) = 0, \quad t \geq t_0 \geq 0, \tag{1.1}$$

where $r \in C([t_0,\infty),(0,\infty))$, $p,q \in C([t_0,\infty),\mathbb{R})$, $f \in C^1(\mathbb{R},\mathbb{R})$ such that $xf(x) > 0$ and $f'(x) \geq K > 0$ for $x \neq 0$, K is a constant. As usual, a nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation of solutions for different classes of second order differential equations [1–12]. Many results have been obtained for particular cases of (1.1), such as the second order linear differential equation

$$x'' + p(t)x' + q(t)x = 0, \quad t \geq t_0 \geq 0. \tag{1.2}$$

It is well known that equation (1.2) can be reduced via suitable Sturm-Liouville transformation to the undamped equation. But, here an additional assumption is added on $p(t)$, that is $p(t)$ assumed to be continuously differentiable. If introducing

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the Sturm-Liouville transformation

$$y(t) = x(t) \exp\left(-\frac{1}{2}\int_{t_0}^{t} p(s)ds\right) \tag{1.3}$$

for the equation (1.2), we have

$$y'' + \left[q(t) - \frac{p^2(t)}{4} - \frac{p'(t)}{2}\right] y = 0, \quad t \geq t_0. \tag{1.4}$$

Oscillation criteria for the undamped differential equation

$$y'' + q(t)y = 0, \quad t \geq t_0, \tag{1.5}$$

and more general differential equation

$$\left(r\left(t\right) y'\right)' + q(t)y = 0, \quad t \geq t_0, \tag{1.6}$$

have been also extensively studied by many authors (see [2], [3], [4], [9] and references cited therein). The known Fite [2], Leighton [4] and Wintner [9] criterion showed that

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\int_{t_0}^{t} q(s)ds = \infty \tag{1.7}$$

was sufficient for equation (1.5) to be oscillatory. Wintner [9] proved that

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} Q(s)ds = \infty \tag{1.8}$$

was also an oscillation criterion for equation (1.5).

Although (1.3) is an oscillation-preserving transformation under the additional assumption that $p(t)$ is differentiable or at least $p(t)$ is absolutely continuous so that $p'(t)$ is defined. But this superfluous condition was not assumed in Sobol's paper (see [7]). By using polar coordinates transformation, he proved that

$$\lim_{t\to\infty}\left[Q(t) - \frac{p(t)}{2} - \frac{1}{4}\int_{t_0}^{t} p^2(s)ds\right] = \infty \tag{1.9}$$

was sufficient for equation (1.2) to be oscillatory. Wong [10] noticed this point and gave several oscillation criteria for equation (1.2), which generalized the results due to Wintner [9] and Kamenev [3].

Recently, Abdullah [1] presented the following two results for the oscillation of equation (1.2) with $p(t) < 0$ on $[t_0, \infty)$.

Theorem A. If $p(t) < 0$ on $[t_0, \infty)$ is such that

$$\int_{t_0}^{\infty}\left[q(s) - \frac{p^2(s)}{4}\right] ds = \infty, \tag{1.10}$$

then equation (1.2) is oscillatory.

Theorem B. If $p(t) < 0$ on $[t_0, \infty)$ and there exists a non vanishing function $g(t) \in C^1([t_0, \infty), (0, \infty))$, such that

$$\int_{t_0}^{\infty} \frac{ds}{g(s)} = \infty \tag{1.11}$$

and

$$\lim_{t\to\infty} \left\{ \int_{t_0}^{t} \left[g(s)q(s) - \frac{p^2(s)g(s)}{4} - \frac{(g'(s))^2}{4g(s)} + \frac{p(s)g'(s)}{2} \right] ds + \frac{g'(t)}{2} \right\} = \infty, \tag{1.12}$$

then equation (1.2) is oscillatory.

Remark 1.1. When $g(t) = 1$, it is easy to see that Theorem B reduces to Theorem A.

More recently, Zheng and Liu [12] also obtain following oscillation results for the equation (1.2). They assume that $g(t) \in C^2([t_0, \infty), (0, \infty))$ is a given function, $h(t) = -\frac{g'(t)}{2g(t)}$ and

$$\Phi(t) = \int_{t_0}^{t} g(s) \left[q(s) - h(s)p(s) + h^2(s) - h'(s) - \frac{p^2(s)}{4} \right] ds - \frac{g(t)p(t)}{2}.$$

Theorem C. Suppose that

$$\int_{t_0}^{\infty} \frac{ds}{g(s)} = \infty \tag{1.13}$$

holds. Then equation (1.2) is oscillatory provided

$$\lim_{t\to\infty} \Phi(t) = \infty. \tag{1.14}$$

Theorem D. Suppose that

$$\int_{t_0}^{\infty} \left(\int_{t_0}^{s} g(\tau) d\tau \right)^{-1} ds = \infty \tag{1.15}$$

holds. Then equation (1.2) is oscillatory provided

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \Phi(s) ds = \infty. \tag{1.16}$$

In this paper, by using a generalized Riccati transformation, we give some new oscillation criteria for equation (1.1). Our results, which extend the oscillation criteria mentioned in the above theorems, aim at developing oscillation criteria for equation (1.1) without any restriction on the signs of $p(t)$ and $q(t)$.

2. MAIN RESULTS

Throughout this paper, we assume that $g(t) \in C^1([t_0,\infty),(0,\infty))$ is a given function, and

$$\Psi(t) = \int_{t_0}^{t}\left[g(s)q(s) - \frac{(g'(s)r(s) - g(s)p(s))^2}{4Kg(s)r(s)}\right]ds + \frac{g'(t)r(t) - g(t)p(t)}{2K}.$$

We are now able to state our main results.

Theorem 2.1. *If*

$$\int_{t_0}^{\infty}\frac{ds}{g(s)r(s)} = \infty \tag{2.1}$$

and

$$\lim_{t\to\infty}\Psi(t) = \infty, \tag{2.2}$$

then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nontrivial nonoscillatory solution of equation (1.1), which, without loss of generality, can be assumed to be $x(t) > 0$, $f(x(t)) > 0$ for $t \geq t_0$. Define

$$w(t) = -g(t)\frac{r(t)x'(t)}{f(x(t))},\ t \geq t_0. \tag{2.3}$$

Then differentiating (2.3) and making use of (1.1), it follows that for all $t \geq t_0$, we obtain

$$w'(t) = \left(\frac{g'(t)}{g(t)} - \frac{p(t)}{r(t)}\right)w(t) + g(t)q(t) + \frac{w^2(t)}{g(t)r(t)}f'(x(t)) \tag{2.4}$$

and using $f'(x) \geq K > 0$ where K is a constant, we get for $t \geq t_0$,

$$w'(t) \geq \tfrac{K}{g(t)r(t)}\left[\left(w(t) + \tfrac{g'(t)r(t)-g(t)p(t)}{2K}\right)^2 - \left(\tfrac{g'(t)r(t)-g(t)p(t)}{2K}\right)^2\right] + g(t)q(t). \tag{2.5}$$

Define $H(t) = w(t) + \frac{g'(t)r(t)-g(t)p(t)}{2K}$, rewrite (2.5), and integrate from t_0 to $t \geq t_0$, we have

$$H(t) \geq w(t_0) + \int_{t_0}^{t}\frac{K}{g(s)r(s)}H^2(s)ds + \Psi(t). \tag{2.6}$$

Now, using (2.2), we can choose t_1 sufficiently large so that

$$H(t) \geq \int_{t_0}^{t}\frac{K}{g(s)r(s)}H^2(s)ds \tag{2.7}$$

holds for $t \geq t_1$. Define a function $R(t)$ for $t \geq t_1$ by

$$R(t) = \int_{t_0}^{t}\frac{K}{g(s)r(s)}H^2(s)ds. \tag{2.8}$$

Thus $H(t) > R(t) > 0$. Differentiating (2.8), we have

$$R'(t) = \frac{K}{g(t)r(t)}H^2(t) > \frac{K}{g(t)r(t)}R^2(t). \tag{2.9}$$

Dividing (2.9) through by $R^2(t)$ and integrating from t_1 to t, we obtain

$$\int_{t_1}^{t} \frac{K}{g(s)r(s)}ds < \frac{1}{R(t_1)} - \frac{1}{R(t)}, \tag{2.10}$$

since $R(t) > 0$, therefore

$$\int_{t_1}^{t} \frac{K}{g(s)r(s)}ds < \frac{1}{R(t_1)}. \tag{2.11}$$

We obtain a desired contradiction with (2.1) as $t \to \infty$. This completes the proof of the theorem. □

If $r(t) = 1$ and $g(t) = 1$ in the above theorem, we have the following result for equation (1.1).

Corollary 2.2. *If*

$$\lim_{t\to\infty} \left\{ \int_{t_0}^{t} \left[q(s) - \frac{p^2(s)}{4K} \right] ds - \frac{p(t)}{2K} \right\} = \infty, \tag{2.12}$$

then equation (1.1) is oscillatory.

Remark 2.3. When $f(x) = x$, Corollary 2.2 reduces to Sobol's result given by (1.9).

Remark 2.4. Let $r(t) = 1$ and $f(x) = x$. If we compare Theorem B (or Theorem A) with Theorem 2.1 (or Theorem 2.1 with $g(t) = 1$) respectively, it is easy to see that the sign condition on $p(t) < 0$ in Theorem B (or Theorem A) can be dropped. We will see this by Example 2.10. Thus, our result is weaker conditions than those of Theorem A or B for equation (1.2).

Theorem 2.5. *If*

$$\int_{t_0}^{\infty} \left(\int_{t_0}^{s} g(\tau)r(\tau)d\tau \right)^{-1} ds = \infty \tag{2.13}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \Psi(s)ds = \infty, \tag{2.14}$$

then equation (1.1) is oscillatory.

Proof. Suppose to the contrary that there is a nontrivial nonoscillatory solution $x(t)$ of equation (1.1). Without loss of generality, we may assume $x(t) > 0$, $f(x(t)) > 0$ for $t \geq t_0$. Then, it follows from the proof of Theorem 2.1, we obtain (2.6). Integrate (2.6) from t_0 to t and divide through by t to obtain

$$\frac{1}{t} \int_{t_0}^{t} H(s)ds \geq w(t_0)\frac{t - t_0}{t} + \frac{1}{t} \int_{t_0}^{t} R(s)ds + \frac{1}{t} \int_{t_0}^{t} \Psi(s)ds. \tag{2.15}$$

By (2.14), we can choose t_1 sufficiently large so that $t \geq t_1$

$$\int_{t_0}^{t} H(s)ds - \int_{t_0}^{t} R(s)ds \geq 0. \tag{2.16}$$

Denote $A(t) = \int_{t_0}^{t} R(s)ds$. Using Hölder inequality, we have

$$\begin{aligned} A^2(t) &\leq \left(\int_{t_0}^{t} H(s)ds\right)^2 = \left(\int_{t_0}^{t} \frac{\sqrt{g(s)r(s)}}{\sqrt{K}} \frac{H(s)\sqrt{K}}{\sqrt{g(s)r(s)}}ds\right)^2 \\ &\leq \left(\int_{t_0}^{t} \frac{g(s)r(s)}{K}ds\right)\left(\int_{t_0}^{t} \frac{K}{g(s)r(s)}H^2(s)ds\right) \\ &= \frac{1}{K}R(t)\left(\int_{t_0}^{t} g(s)r(s)ds\right) \\ &= \frac{1}{K}A'(t)\left(\int_{t_0}^{t} g(s)r(s)ds\right). \end{aligned} \tag{2.17}$$

Dividing (2.17) through by $\frac{A^2(t)}{K}\left(\int_{t_0}^{t} g(s)r(s)ds\right)$ and integrating from t_1 to t, we obtain

$$K\int_{t_1}^{t}\left(\int_{t_0}^{s} g(\tau)r(\tau)d\tau\right)^{-1} ds \leq \frac{1}{A(t_1)} - \frac{1}{A(t)} \leq \frac{1}{A(t_1)}. \tag{2.18}$$

But (2.18) incompatible with (2.13) as $t \to \infty$. This completes the proof of Theorem 2.5. □

Remark 2.6. Let $r(t) = 1$ and $f(x) = x$. Although the condition on $g(t) \in C^2([t_0,\infty),(0,\infty))$ given in Theorems C and D, our results just depend on $g(t) \in C^1([t_0,\infty),(0,\infty))$ in Theorems 2.1 and 2.5, respectively. Thus, our results are sharper than those of Theorem C or D for equation (1.2).

Remark 2.7. Theorem 2.1 generalizes and extends Theorem A or B for the nonlinear equation (1.1). Moreover, Theorems 2.1 and 2.5 also generalize and extend Theorems C and D for the nonlinear equation (1.1), respectively.

Remark 2.8. The results in this paper are still true if we replace condition $f'(x) \geq K > 0$ for $x \neq 0$ with the following one

$$\frac{f(x)}{x} \geq K > 0 \quad \text{for } x \neq 0.$$

But $q(t)$ should be nonnegative in this case.

Finally, we give some examples to illustrate the efficiency and applicability of our results. These examples are not covered by any of the results of Abdullah [1] and Zheng and Liu [12].

Example 2.9. Consider the nonlinear differential equation

$$\left(t^2x'\right)' + at\, x' + bf(x) = 0, \qquad t \geq 1, \tag{2.19}$$

where a, b are two real constants with $b > 0$ and $f(x)$ is any function which satisfies $xf(x) > 0$ and $f'(x) \geq K > 0$ for $x \neq 0$, K is a constant. To show the applicability of Theorem 2.1, choose $g(t) = \frac{1}{t}$. It is clear that condition (2.1) is satisfied. Condition (2.2) is satisfied as follows:

$$\begin{aligned}\lim_{t\to\infty}\Psi(t) &= \lim_{t\to\infty}\left\{\int_{t_0}^{t}\left[g(s)q(s)-\tfrac{(g'(s)r(s)-g(s)p(s))^2}{4Kg(s)r(s)}\right]ds+\tfrac{g'(t)r(t)-g(t)p(t)}{2K}\right\}\\ &= \lim_{t\to\infty}\left\{\int_{1}^{t}\left[\frac{b}{s}-\frac{(a+1)^2}{4Ks}\right]ds-\frac{a+1}{2K}\right\}\\ &= \lim_{t\to\infty}\left\{\frac{4Kb-(a+1)^2}{4K}\ln t-\frac{a+1}{2K}\right\}\\ &= \infty\ , \quad \text{if} \quad (a+1)^2<4Kb.\end{aligned}$$

Hence, equation (2.19) is oscillatory for $(a+1)^2 < 4Kb$. In particular, equation (2.19) with $a = \alpha - 2$, $b = \beta > 0$ and $f(x) = x$ is also oscillatory for $(\alpha - 1)^2 < 4\beta$. On the other hand, if $(\alpha - 1)^2 \geq 4\beta$, evidently, equation (2.19) has a nonoscillatory solution $x(t) = t^{\frac{1-\alpha+\sqrt{(\alpha-1)^2-4\beta}}{2}}$. At this time, under the additional assumption that $r(t)$ is differentiable, this example reduces to Zheng and Liu [12]'s example. Moreover, equation (2.19) with $a = -1$, $b = 1$ and $f(x) = x$ is oscillatory. This fact is directly verified by noting that all the solution of equation (2.19) is given by $x(t) = c_1 \sin(\ln t) + c_2 \cos(\ln t)$ where c_1, c_2 are two real constants with $c_1^2 + c_2^2 \neq 0$.

Example 2.10. Consider the nonlinear differential equation

$$x'' + \frac{A}{t}\,x' + \left(1+\frac{B}{t^2}\right)\,f(x) = 0, \qquad t \geq 1, \tag{2.20}$$

where A, B are two real constants, and $f(x)$ is any function which satisfies $xf(x) > 0$ and $f'(x) \geq K > 0$ for $x \neq 0$, K is a constant. To show the applicability of Theorem 2.1, choose $g(t) = t$. It is clear that condition (2.1) is satisfied. Condition (2.2) is satisfied as follows:

$$\begin{aligned}\lim_{t\to\infty}\Psi(t) &= \lim_{t\to\infty}\left\{\int_{t_0}^{t}\left[g(s)q(s)-\tfrac{(g'(s)r(s)-g(s)p(s))^2}{4Kg(s)r(s)}\right]ds+\tfrac{g'(t)r(t)-g(t)p(t)}{2K}\right\}\\ &= \lim_{t\to\infty}\left\{\int_{1}^{t}\left[s+\frac{B}{s}-\frac{(A-1)^2}{4Ks}\right]ds+\frac{1-A}{2K}\right\}\\ &= \lim_{t\to\infty}\left\{\frac{t^2}{2}+\frac{4KB-(A-1)^2}{4K}\ln t+\frac{1-A}{2K}-\frac{1}{2}\right\}\\ &= \infty\ .\end{aligned}$$

So, equation (2.20) is oscillatory by virtue of Theorem 2.1. Moreover, if we choose $g(t) = 1$ then it is easy to see that condition (2.12) is satisfied. So, equation (2.20) is oscillatory by Corollary 2.2. In particular, equation (2.20) with $A = -2$, $B = 2$ and $f(x) = x$ is also oscillatory. At this time, this example reduces to Abdullah [1]'s example. This fact is directly verified by noting that all the solution of equation

(2.20) is given by $x(t) = t\,(c_1 \sin t + c_2 \cos t)$ where c_1, c_2 are two real constants with $c_1^2 + c_2^2 \neq 0$.

Furthermore, we can choose $g(t) = 1$ to show the applicability of Theorem 2.5 for equation (2.20). It is easy to see that condition (2.13) is satisfied. Condition (2.14) is satisfied as follows:

$$\begin{aligned}\Psi(t) &= \int_{t_0}^{t}\left[g(s)q(s) - \frac{(g'(s)r(s) - g(s)p(s))^2}{4Kg(s)r(s)}\right]ds + \frac{g'(t)r(t) - g(t)p(t)}{2K}\\ &= \int_{1}^{t}\left[1 + \frac{B}{s^2} - \frac{A^2}{4Ks^2}\right]ds - \frac{A}{2Kt}\\ &= t + \frac{A^2 - 2A - 4KB}{4Kt} + \frac{4KB - 4K - A^2}{4K},\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\Psi(s)ds &= \lim_{t\to\infty}\frac{1}{t}\int_{1}^{t}\left[s + \frac{A^2 - 2A - 4KB}{4Ks} + \frac{4KB - 4K - A^2}{4K}\right]ds\\ &= \lim_{t\to\infty}\frac{1}{t}\left[\tfrac{t^2}{2} + \tfrac{A^2-2A-4KB}{4K}\ln t + \tfrac{4KB-4K-A^2}{4K}t - \tfrac{1}{2} - \tfrac{4KB-4K-A^2}{4K}\right]\\ &= \infty\ .\end{aligned}$$

Therefore, equation (2.20) is oscillatory by Theorem 2.5.

Finally, we give a simple example where Theorem 2.5 applies, but Theorem 2.1 does not.

Example 2.11. Consider the nonlinear differential equation

$$\left[\left(\frac{3e^{t} - e^{-t}}{2}\right)x'\right]' - e^{-t}x' + \left(e^{2t} - 1\right)f\left(x\right) = 0, \qquad t \geq 0, \tag{2.21}$$

where $f(x)$ is any function which satisfies $xf(x) > 0$ and $f'(x) \geq K > 0$ for $x \neq 0$, K is a constant. It is easy to see that if we choose $g(t) = \frac{2e^{2t}}{3e^{2t}-1}$ then Theorem 2.1 cannot be applied to the oscillation of equation (2.21), because of the condition (2.1) is not satisfied. But, we can prove the oscillatory character of equation (2.21) by using Theorem 2.5. Condition (2.13) is satisfied as follows:

$$\int_{t_0}^{\infty}\left(\int_{t_0}^{s}g(\tau)r(\tau)d\tau\right)^{-1}ds = \int_{0}^{\infty}\left(\int_{0}^{s}e^{\tau}\,d\tau\right)^{-1}ds = \int_{0}^{\infty}\frac{ds}{e^{s} - 1} = \infty.$$

And, condition (2.14) is also satisfied as follows:

$$\begin{aligned}\Psi(t) &= \int_{t_0}^{t}\left[g(s)q(s) - \frac{(g'(s)r(s) - g(s)p(s))^2}{4Kg(s)r(s)}\right]ds + \frac{g'(t)r(t) - g(t)p(t)}{2K}\\ &= \int_{0}^{t}\frac{2}{3 - e^{-2s}}\left(e^{2s} - 1\right)ds = \frac{1}{3}e^{2t} - \frac{2}{9}\ln(3e^{2t} - 1) - \frac{1}{3} + \frac{2}{9}\ln 2,\end{aligned}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\Psi(s)ds = \lim_{t\to\infty}\frac{1}{t}\int_{0}^{t}\left[\frac{1}{3}e^{2s} - \frac{2}{9}\ln(3e^{2s} - 1) - \frac{1}{3} + \frac{2}{9}\ln 2\right]ds$$

$$= \lim_{t\to\infty} \frac{1}{t} \left\{ \tfrac{1}{6}\left(e^{2t}-1\right) - \tfrac{2}{9}\int_0^t \ln(3e^{2s}-1)ds + \left(-\tfrac{1}{3}+\tfrac{2}{9}\ln 2\right)t \right\}$$
$$= \infty \ .$$

Hence, equation (2.21) is oscillatory by virtue of Theorem 2.5.

Note that if we take $f(x) = x(K + 1 + \cos x)$ in the equation (2.19), (2.20) or (2.21), there is no K for which $f'(x) \geq K > 0$ for $x \neq 0$, but the function f satisfies $\frac{f(x)}{x} \geq K > 0$ for $x \neq 0$. So, in the above examples, we may take equation (2.20) with $B \geq 0$ in this case.

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