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COMPLETE CONTROLLABILITY OF STOCHASTIC INTEGRODIFFERENTIAL SYSTEMS

K. BALACHANDRAN, J.-H.KIM, AND S.KARTHIKEYAN

Department of Mathematics, Bharathiar University, Coimbatore 641 046, India
Department of Mathematics, Yonsei University, Seoul 120-749, Korea

ABSTRACT. In this paper sufficient conditions for the complete controllability of stochastic semilinear integrodifferential system in finite dimensional spaces are established. The results are obtained by using the Banach fixed point theorem. An example is provided to illustrate the technique.
KEYWORDS: Complete controllability, stochastic integrodifferential system, Banach fixed point theorem.

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1. INTRODUCTION

The problem of controllability of linear deterministic system is well documented. It is well known that controllability of deterministic equations are widely used in analysis and the design of control system. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals. In many dynamical systems, it is possible to steer the dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls; that is, there are systems which are completely controllable. If the system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability.

The problem of controllability of linear stochastic systems of the form

$$(1)\qquad \begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + \tilde{\sigma}(t)dw(t) \\ x(0) \;= x_0, \quad t \in [0, T] \end{cases}$$

has been studied by various authors[4, 8, 11, 16] where A, B are matrices of dimensions $n \times n$ and $n \times m$ respectively and $\tilde{\sigma} : [0, T] \to \mathbb{R}^{n\times n}$.

The controllability of nonlinear deterministic systems in finite dimensional space has been extensively studied by several authors, see[1, 2, 5] and references therein. There are very few works about the controllability of nonlinear stochastic systems. In [15], the authors formulated stochastic ϵ-controllability and controllability with probability one for a class of stochastic nonlinear systems. Sunahara et al[14] obtained sufficient conditions for the stochastic controllability and observability of nonlinear

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systems by using stochastic Liapunov method. Klamka and Socha[6, 7] discussed the stochastic ϵ-controllability of continuous nonlinear stochastic dynamical systems by the same technique. Dauer and Balachandran[3] studied the sample controllability of stochastic nonlinear systems by means of fixed point principle.

Mahmudov[9, 10] studied approximate controllability of non-linear stochastic system when nonlinear terms are uniformly bounded and satisfy the Lipschitz condition. Recently, Mahmudov and Zorlu[13] investigated the approximate and complete controllability of general nonlinear stochastic system by using the Picard type iteration technique. In this paper we examine the controllability of a semilinear stochastic integrodifferential system

$$\begin{cases} dx(t) = \Big[Ax(t) + Bu(t) + f(t, x(t)) + \displaystyle\int_0^t g(t, s, x(s))ds\Big]dt \\ \qquad\qquad +\sigma(t, x(t))dw(t) \\ x(0) \ = x_0, \quad t \in [0, T] \end{cases} \tag{2}$$

where A and B are matrices of dimensions $n \times n$ and $n \times m$ respectively, $g : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $u(t)$ is a feedback control and w is an n-dimensional Wiener process. Here we show that the complete controllability of the semilinear stochastic integrodifferential system under the natural assumption that the associated linear control system is completely controllable.

2. PRELIMINARIES

In this paper we use the following notations:

- $(\Omega, \mathcal{F}, P) :=$ the probability space with probability measure P on Ω.
- $\{\mathcal{F}_t | t \in [0, T]\} :=$ the filtration generated by $\{w(s) : 0 \le s \le t\}$.
- $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) :=$ the Hilbert space of all $\mathcal{F}_T$-measurable square integrable variables with values in $\mathbb{R}^n$.
- $L_p^{\mathcal{F}}([0, T], \mathbb{R}^n) :=$ the Banach space of all p-integrable and $\mathcal{F}_t$-measurable processes with values in $\mathbb{R}^n$, for $p \ge 2$.
- $\mathcal{B}_2 :=$ the Banach space of all square integrable and $\mathcal{F}_t$-adapted processes $\varphi(t)$ with norm

$$\|\varphi\|^2 := \sup_{t\in[0,T]} \mathbf{E}\|\varphi(t)\|^2.$$

- $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) :=$ The space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- $\Phi(t) := \exp(At)$ and $U_{ad} := L_2^{\mathcal{F}}([0, T], \mathbb{R}^m)$.

Now let us introduce the following matrices and sets.

1. The operator $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}([0, T], \mathbb{R}^m), L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$ is defined by

$$L_0^T u = \int_0^T \Phi(T - s)Bu(s)ds.$$

and set of all states attainable from x_0 in time $t > 0$.

$$\mathcal{R}_t(x_0) = \{x(t; x_0, u) : u(\cdot) \in L_2^{\mathcal{F}}([0,T], \mathbb{R}^m)\}$$

where $x(t; x_0, u)$ is the solution of (2) corresponding to $x_0 \in R^n$ and $u(\cdot) \in L_2^{\mathcal{F}}([0,T], \mathbb{R}^m)$.

Clearly the adjoint $(L_0^T)^* : L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \to L_2^{\mathcal{F}}([0,T], \mathbb{R}^m)$ is defined by

$$[(L_0^T)^* z](t) = B^*\Phi^*(T-t)\mathbf{E}\{z \mid \mathcal{F}_t\}$$

2. The controllability operator Π_0^T associated with (1) is defined by

$$\Pi_0^T\{\cdot\} = L_0^T(L_0^T)^*\{\cdot\} = \int_0^T \Phi(T-t)BB^*\Phi^*(T-t)\mathbf{E}\{\cdot \mid \mathcal{F}_t\}dt$$

which belongs to $\mathcal{L}(L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n), L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$ and the controllability matrix $\Gamma_s^T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is

$$\Gamma_s^T = \int_s^T \Phi(T-t)BB^*\Phi^*(T-t)dt, \quad 0 \le s < t$$

Definition 2.1. The stochastic system (2) is completely controllable on $[0,T]$ if

$$\mathcal{R}_T(x_0) = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

that is, all the points in $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ can be reached from the point x_0 at time T.

The following lemma whose proof can be found in [12] give a formula for a control that steers the system from initial state x_0 to an arbitrary final state x_T.

Lemma 2.1. Assume that the operator Π_0^T is invertible. Then for arbitrary $x_T \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$, $f(\cdot) \in L_2^{\mathcal{F}}([0,T], \mathbb{R}^n)$, $\sigma(\cdot) \in L_2^{\mathcal{F}}([0,T], \mathbb{R}^{n\times n})$, the control

$$\begin{aligned} u(t) &= B^*\Phi^*(T-t)\mathbf{E}\Big\{(\Pi_0^T)^{-1}\Big(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-s)f(s)ds \\ &\quad - \int_0^T \Phi(T-s)\tilde{\sigma}(s)dw(s) - \int_0^T \Phi(T-s)\left(\int_0^s g(s,\tau)d\tau\right)ds\Big) \,\Big|\, \mathcal{F}_t\Big\} \end{aligned} \tag{3}$$

transfers the system

$$\begin{aligned} x(t) &= \Phi(t)x_0 + \int_0^t \Phi(t-s)[Bu(s) + f(s)]ds + \int_0^t \Phi(t-s)\tilde{\sigma}(s)dw(s) \\ &\quad + \int_0^t \Phi(t-s)\left(\int_0^s g(s,\tau)d\tau\right)ds \end{aligned} \tag{4}$$

from $x_0 \in \mathbb{R}^n$ to x_T at time T and

$$\begin{aligned} x(t) &= \Phi(t)x_0 + \Pi_0^t\Big[\Phi^*(T-t)(\Pi_0^T)^{-1}\Big(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-r)f(r)dr \\ &\quad - \int_0^T \Phi(T-r)\tilde{\sigma}(r)dw(r) - \int_0^T \Phi(T-r)\left(\int_0^r g(r,\tau)d\tau\right)dr\Big)\Big] \\ &\quad + \int_0^t \Phi(t-s)f(s)ds + \int_0^t \Phi(t-s)\tilde{\sigma}(s)dw(s) + \int_0^t \Phi(t-s)\left(\int_0^s g(s,\tau)d\tau\right)ds \end{aligned} \tag{5}$$

3. CONTROLLABILITY RESULTS

In this section, we derive controllability conditions for the semilinear stochastic integrodifferential system (2) by using the contraction mapping principle.

We impose the following conditions on data of the problem:

(H1) The functions f, g and σ satisfies the Lipschitz condition and there exist constants $L_1, L_2 > 0$ for $x_1, x_2 \in \mathbb{R}^n$ and $0 \leq s < t \leq T$ such that

$$\|f(t,x_1) - f(t,x_2)\|^2 + \|\sigma(t,x_1) - \sigma(t,x_2)\|^2 \leq L_1\|x_1 - x_2\|^2$$

$$\| \int_0^t [g(t,s,x_1(s)) - g(t,s,x_2(s))]ds\|^2 \leq L_2\|x_1 - x_2\|^2$$

(H2) The functions f, g and σ are continuous and satisfies the usual linear growth condition, that is, there exists a constant $L > 0$ such that

$$\|f(t,x)\|^2 + \| \int_0^t g(t,s,x(s))ds\|^2 + \|\sigma(t,x)\|^2 \leq L(\|x\|^2 + 1)$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$.

(H3) The linear system (1) is completely controllable.

Remark 3.1. In [8] and [11], it is shown that the complete controllability and the approximate controllability of the linear system (1) coincide. But in the case of semilinear stochastic systems they differ. That is why we study the approximate and complete controllability of the semilinear stochastic integrodifferential system (2) separately.

By a solution of the system (2), we mean a solution of the nonlinear integral equation

$$\begin{aligned} x(t) &= \Phi(t)x_0 + \int_0^t \Phi(t-s)[Bu(s) + f(s,x(s))]ds + \int_0^t \Phi(t-s)\sigma(s,x(s))dw(s) \\ &\quad + \int_0^t \Phi(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \end{aligned} \tag{6}$$

It is obvious that under the conditions (H1) and (H2), for every $u(\cdot) \in U_{ad}$ the integral equation (6) has a unique solution in $\mathcal{B}_2$. To apply the contraction mapping principle, we define the nonlinear operator $\mathcal{G}$ from $\mathcal{B}_2$ to $\mathcal{B}_2$ as follows:

$$\begin{aligned} (\mathcal{G}x)(t) &= \Phi(t)x_0 + \int_0^t \Phi(t-s)[Bu(s) + f(s,x(s))]ds + \int_0^t \Phi(t-s)\sigma(s,x(s))dw(s) \\ &\quad + \int_0^t \Phi(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \end{aligned} \tag{7}$$

where

$$\begin{aligned} u(t) &= B^*\Phi^*(T-t)\mathbf{E}\Big\{(\Pi_0^T)^{-1}\Big(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-s)f(s,x(s))ds \\ &\quad - \int_0^T \Phi(T-s)\sigma(s,x(s))dw(s) - \int_0^T \Phi(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds\Big)\Big|\mathcal{F}_t\Big\} \end{aligned} \tag{8}$$

From Lemma 2.1, the control (8) transfers the system (6) from the initial state x_0 to the final state x_T provided that the operator $\mathcal{G}$ has a fixed point. So, if the operator $\mathcal{G}$ has a fixed point then the system (6) is completely controllable. Now for our convenience, let us introduce the following notations

$$l_1 = \max\{\|\Phi(t)\|^2 : t \in [0,T]\}, \qquad l_2 = \|B\|^2,$$

$$l_3 = \mathbf{E}\|x_T\|^2, \qquad M = \max\{\|\Gamma_s^T\|^2 : s \in [0,T]\}$$

Note that if the assumption (H3) holds, then for some $\gamma > 0$,

$$\mathbf{E}\left\langle \Pi_0^T z, z\right\rangle \geq \gamma \mathbf{E}\|z\|^2, \qquad \text{for all } z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

see [9] and consequently, $\|(\Pi_0^T)^{-1}\| \leq \dfrac{1}{\gamma} = l_4$.

Theorem 3.1. Assume that the conditions (H1),(H2) and (H3) hold. If the inequality

$$6l_1(1 + Ml_1l_4)(L_1 + L_2)(T+1)T < 1 \tag{9}$$

is satisfied, then the system (6) is completely controllable.

Proof. To prove the complete controllability it is enough to show that $\mathcal{G}$ has a fixed point in $\mathcal{B}_2$. To do this, we use the contraction mapping principle. To apply the principle, first we show that $\mathcal{G}$ maps $\mathcal{B}_2$ into itself.

Now by Lemma 2.1, we have

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{G}x)(t)\|^2 \\
&= \mathbf{E}\Big\|\Phi(t)x_0 + \Pi_0^t\Big[\Phi^*(T-t)(\Pi_0^T)^{-1}\Big(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-r)f(r,x(r))dr \\
&\quad - \int_0^T \Phi(T-r)\sigma(r,x(r))dw(r) - \int_0^T \Phi(T-r)\Big[\int_0^r g(r,\tau,x(\tau))d\tau\Big]dr\Big)\Big] \\
&\quad + \int_0^t \Phi(t-s)f(s,x(s))]ds + \int_0^t \Phi(t-s)\sigma(s,x(s))dw(s) \\
&\quad + \int_0^t \Phi(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds\Big\|^2 \\
&\leq 5l_1\mathbf{E}\|x_0\|^2 + 5Ml_1l_4\mathbf{E}\Big\|x_T - \Phi(T)x_0 - \int_0^T \Phi(T-r)f(r,x(r))dr \\
&\quad - \int_0^T \Phi(T-r)\sigma(r,x(r))dw(r) - \int_0^T \Phi(T-r)\Big[\int_0^r g(r,\tau,x(\tau))d\tau\Big]dr\Big]\Big\|^2 \\
&\quad + 5l_1T\mathbf{E}\int_0^t \|f(s,x(s))\|^2 ds + 5l_1\mathbf{E}\int_0^t \|\sigma(s,x(s))\|^2 ds \\
&\quad + 5l_1T\mathbf{E}\int_0^t \Big\|\int_0^s g(s,\tau,x(\tau))d\tau\Big\|^2 ds \\
&\leq 5l_1\mathbf{E}\|x_0\|^2 + 25Ml_1l_4\Big\{l_3 + l_1\|x_0\|^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+5l_1(T+1)L\int_0^t(1+\mathbf{E}\|x(r)\|^2)dr+25Ml_1^2l_4(T+1)L\int_0^t(1+\mathbf{E}\|x(r)\|^2)dr\\
\leq\ & 5l_1\|x_0\|^2+25Ml_1l_4\Big\{l_3+l_1\|x_0\|^2\Big\}\\
&+(5l_1+25Ml_1^2l_4)(T+1)L\int_0^t(1+\mathbf{E}\|x(r)\|^2)dr \qquad (10)
\end{aligned}
$$

It follows from (10) and the condition (H2), there exists $C_1 > 0$ such that

$$
\begin{aligned}
\mathbf{E}\|(\mathcal{G}x)(t)\|^2 &\leq C_1\Big(1+\int_0^T\mathbf{E}\|x(r)\|^2dr\Big)\\
&\leq C_1\Big(1+T\sup_{0\leq r\leq T}\mathbf{E}\|x(r)\|^2\Big)
\end{aligned}
$$

for all $t\in[0,T]$. Therefore $\mathcal{G}$ maps $\mathcal{B}_2$ into itself.
Secondly, we show that $\mathcal{G}$ is a contraction mapping.

$$
\begin{aligned}
&\mathbf{E}\|(\mathcal{G}x_1)(t)-\mathcal{G}x_2)(t))\|^2\\
&\quad= \mathbf{E}\Big\|\int_0^t\Phi(t-s)[f(s,x_1(s))-f(s,x_2(s))]ds\\
&\qquad+\int_0^t\Phi(t-s)[\sigma(s,x_1(s))-\sigma(s,x_2(s))]dw(s)\\
&\qquad+\int_0^t\Phi(t-s)\Big[\int_0^s[g(s,\tau,x_1(\tau))-g(s,\tau,x_2(\tau))]d\tau\Big]ds\\
&\qquad+\Pi_0^T\Phi^*(T-t)(\Pi_0^T)^{-1}\Big(\int_0^T\Phi(T-r)[f(s,x_2(s))-f(s,x_1(s))]ds\\
&\qquad+\int_0^T\Phi(T-s)[\sigma(s,x_2(s))-\sigma(s,x_1(s))]ds\\
&\qquad+\int_0^T\Phi(T-s)\Big[\int_0^r(g(s,\tau,x_2(\tau))-g(s,\tau,x_1(\tau)))d\tau\Big]ds\Big)\Big\|^2\\
&\quad\leq 6l_1(L_1+L_2)(T+1)\mathbf{E}\int_0^t\|x_1(s)-x_2(s)\|^2ds\\
&\qquad+6Ml_1^2l_4(L_1+L_2)(T+1)\mathbf{E}\int_0^t\|x_1(s)-x_2(s)\|^2ds\\
&\quad\leq 6l_1(L_1+L_2)(1+Ml_1l_4)(T+1)\mathbf{E}\int_0^t\|x_1(s)-x_2(s)\|^2ds
\end{aligned}
$$

It results that

$$
\begin{aligned}
&\sup_{t\in[0,t]}\mathbf{E}\|(\mathcal{G}x_1)(t)-\mathcal{G}x_2)(t))\|^2\\
&\qquad\leq 6l_1(L_1+L_2)(1+Ml_1l_4)(T+1)T\sup_{t\in[0,t]}\mathbf{E}\|x_1(t)-x_2(t)\|^2
\end{aligned}
$$

Therefore by (9), $\mathcal{G}$ is a contraction mapping. Then the mapping $\mathcal{G}$ has a unique fixed point $x(\cdot)\in\mathcal{B}_2$ which is the solution of the equation (6). Thus the system (6) is completely controllable.

Remark 3.2. Obviously hypothesis (9) is fulfilled if $L_1 + L_2$ is sufficiently small.

Remark 3.3. Consider the time varying semilinear stochastic integrodifferential system of the form

$$\tag{11} \begin{cases} dx(t) = \Big[A(t)x(t) + B(t)u(t) + f(t, x(t)) + \int_{t_0}^{t} g(t, s, x(s))ds)\Big]dt \\ \qquad\qquad +\sigma(t, x(t))dw(t) \\ x(t_0) = x_0, \quad t \in [t_0, T] \end{cases}$$

where $A(t)$ and $B(t)$ are matrices of dimensions $n \times n$ *and* $n \times m$ respectively and $f,\ g,\ \sigma, w$ are as before.

The solution of the above equation is

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)[Bu(s) + f(s, x(s))]ds + \int_{t_0}^{t} \Phi(t, s)\sigma(s, x(s))dw(s) \\ &\quad + \int_{t_0}^{t} \Phi(t, s)\Big[\int_{t_0}^{s} g(s, \tau, x(\tau))d\tau\Big]ds \end{aligned}$$

where $\Phi(t, t_0)$ is the fundamental matrix of the homogeneous equation $\dot{x}(t) = A(t)x(t)$ with $x(t_0) = x_0$. If the functions f, g and σ satisfy the local Lipschitz condition then a suitable function will steer the system (11) from x_0 to x_T provided the above equation is satisfied.

Remark 3.4. Consider the stochastic integrodifferential system of the form

$$\tag{12} \begin{cases} d[x(t) - h(t, x(t))] = \Big[Ax(t) + Bu(t) + \int_{0}^{t} g(t, s, x(s))ds)\Big]dt \\ \qquad\qquad\qquad\qquad +\sigma(t, x(t))dw(t) \\ x(0) = x_0, \quad t \in [0, T] \end{cases}$$

where A, B, f, g, σ, w are as before and $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in t. The solution of the above equation is

$$\begin{aligned} x(t) &= \Phi(t)[x_0 - h(0, x_0)] + h(t, x(t)) + \int_{0}^{t} A\Phi(t - s)h(s, x(s))ds \\ &\quad + \int_{0}^{t} \Phi(t - s)Bu(s)ds + \int_{0}^{t} \Phi(t - s)\Big[\int_{0}^{s} g(s, \tau, x(\tau))d\tau\Big]ds \\ &\quad + \int_{0}^{t} \Phi(t - s)\sigma(s, x(s))dw(s) \end{aligned} \tag{13}$$

(H4) The functions g, h and σ satisfies the Lipschitz condition and there exist constants $L_1, L_2, L_3 > 0$ for $x_1, x_2 \in \mathbb{R}^n$ and $0 \le s \le t \le T$ such that

$$\|\sigma(t, x_1) - \sigma(t, x_2)\|^2 \le L_1\|x_1 - x_2\|^2$$

$$\|\int_{0}^{t} [g(t, s, x_1(s)) - g(t, s, x_2(s))]ds\|^2 \le L_2\|x_1 - x_2\|^2$$

$$\|h(t, x_1) - h(t, x_2)\|^2 \le L_3\|x_1 - x_2\|^2$$

(H5) The functions h, g and σ are continuous and there exists a constant $L > 0$ such that

$$\|\int_0^t g(t,s,x(s))ds\|^2 + \|\sigma(t,x)\|^2 \leq L(\|x\|^2 + 1)$$

$$\|h(t,x)\|^2 \leq L(\|x\|^2 + 1)$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$.

Clearly, under the conditions (H4) and (H5), for every $u(\cdot) \in U_{ad}$ the integral equation (13) has a unique solution in $\mathcal{B}_2$. To apply the contraction mapping principle, we define the nonlinear operator $\mathcal{S}$ from $\mathcal{B}_2$ to $\mathcal{B}_2$ as follows:

$$\begin{aligned}(\mathcal{S}x)(t) &= \Phi(t)[x_0 - h(0,x_0)] + h(t,x(t)) + \int_0^t \Phi(t-s)Bu(s)ds \\ &\quad + \int_0^t \Phi(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds + \int_0^t A\Phi(t-s)h(s,x(s))ds \\ &\quad + \int_0^t \Phi(t-s)\sigma(s,x(s))dw(s)\end{aligned}$$

where

$$\begin{aligned}u(t) &= B^*\Phi^*(T-t)\mathbf{E}\Big\{(\Pi_0^T)^{-1}\Big(x_T - \Phi(T)[x_0 - h(0,x_0)] - h(T,x(T)) \\ &\quad - \int_0^T \Phi(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds - \int_0^T \Phi(T-s)\sigma(s,x(s))dw(s) \\ &\quad - \int_0^T A\Phi(T-s)h(s,x(s))ds\Big)\Big|\mathcal{F}_t\Big\}\end{aligned}$$

Clearly the above control transfers the system (12) from the initial state x_0 to the final state x_T provided that the operator $\mathcal{S}$ has a fixed point. So, if the operator $\mathcal{S}$ has a fixed point then the system (12) is completely controllable.

Theorem 3.2. If the conditions (H3)-(H5) hold, then the system (12) is completely controllable provided

$$[7l_1(1 + Ml_1l_4)(T\|A\| + 1)(L_1 + L_2 + L_3) + 7L_3]T < 1. \tag{14}$$

Proof. The proof of this theorem is similar to that of Theorem 3.1 and hence it is omitted.

4. EXAMPLE

Consider a two dimensional nonlinear stochastic integrodifferential control system

$$\begin{cases} dx(t) = \Big[A(t)x(t) + B(t)u(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds\Big]dt \\ \qquad\qquad +\sigma(t,x(t))dw(t) \\ x(0) = x_0 \in \mathbb{R}^2, \quad t \in [0,T] \end{cases} \tag{15}$$

where $w(t)$ is one-dimensional Browninan motion and

$$A(t) = \begin{bmatrix} 0 & e^{-t} \\ 0 & e^{-t} \end{bmatrix}, \quad B(t) = \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

$$f(t, x(t)) = \begin{bmatrix} (2 + \cos x_2(t))x_1(t) + 3x_2(t) \\ \\ (3 + \sin x_1(t))x_2(t) + 2x_1(t) \end{bmatrix}, \quad \sigma(t, x(t)) = \begin{bmatrix} \frac{(2t^2+1)e^{-t}}{(1+x_1(t))} & 0 \\ 0 & \frac{\sin t \ \cos t \ e^{-t}}{(1+x_2(t))} \end{bmatrix}$$

$$\int_0^t g(t, s, x(s))ds = \begin{bmatrix} \int_0^t e^{-x_1(s)}ds \\ \\ \int_0^t e^{-t}(5x_1(s) + 3x_2(s))ds \end{bmatrix}$$

for $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. Take the final point $x_T \in \mathbb{R}^2$. For this system, the fundamental matrix is given by

$$\Phi(t, 0) = \begin{bmatrix} 1 & \exp(1 - e^{-t}) - 1 \\ \\ 0 & \exp(1 - e^{-t}) \end{bmatrix}$$

and the controllability matrix

$$\begin{aligned} \Gamma_0^T &= \int_0^T \Phi(T, t)B(t)B^*(t)\Phi^*(T, t)dt \\ &= \frac{1}{2} \begin{bmatrix} 7 + e^{2-2e^{-T}} - 4e^{-T} - 4e^{1-\cosh(T)+\sinh[T]} & e^{-2e^{-T}}(e - e^{e^{-T}})^2 \\ \\ e^{-2e^{-T}}(e - e^{e^{-T}})^2 & e^{2-2e^{-T}} - 1 \end{bmatrix} \end{aligned}$$

is nonsingular if $T > 0$. Moreover, it is easy to show that for all $x \in \mathbb{R}^2$, $\|f(t, x(t))\|^2 \leq 13x_1^2 + 34\|x_1x_2\| + 25x_2^2 \leq 42\|x\|^2$, $\left\| \int_0^t g(t, s, x(s))ds \right\|^2 \leq 75(T + 1)(1 + \|x\|^2)$ and $\|\sigma(t, x(t))\| \leq 2(2t^2 + 1)e^{-t}$. By choosing T small enough, one can see that the inequality (9) hold and all other conditions stated in Theorem 3.1 are satisfied. Hence, the system (15) is completely controllable on $[0, T]$, that is, the system (15) can be steered from x_0 to x_T.

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