Dynamic Systems and Applications 16 (2007) 673-680

MULTIPLE SOLUTIONS FOR A DIRICHLET PROBLEM INVOLVING THE P-LAPLACIAN

F. CAMMAROTO, A. CHINNÌ, AND B. DI BELLA

Department of Mathematics, University of Messina
98166 Sant'Agata-Messina, Italy

ABSTRACT. In this paper, we establish some multiplicity results for a Dirichlet problem related to a parametric equation involving the p-Laplacian operator. To this aim we make use of a recent local minima result of B. Ricceri.

1. INTRODUCTION

Here and in the sequel Ω is a non-empty bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $p > N$. We are interested in the multiplicity of weak solutions of the following Dirichlet problem

$$(D_{\lambda,\mu}) \qquad \begin{cases} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

where $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions, λ, μ are two positive parameters and $\Delta_p = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

In [2], making use of a three critical points theorem of Ricceri ([6], Theorem 1), the authors established the existence of three weak solutions for the problem $(D_{\lambda,\mu})$ in the case $\mu = 0$. Still in the case $\mu = 0$ but with f depending only on u and having discontinuous nonlinearities, a multiplicity result for $(D_{\lambda,\mu})$ is obtained in [1]. Recently, in [3], the autonomous case of the problem $(D_{\lambda,\mu})$ when $p = 2$ and $N = 1$ has been studied. Here, thanks to a recent result of Ricceri, we will obtain two (or three) solutions of $(D_{\lambda,\mu})$ when $\mu \neq 0$.

Now, we recall the Ricceri's results that will be used in our arguments.

Proposition 1.1 ([5], Proposition 3.1)**.** *Let X be a non-empty set, and Φ, J two real functions on X. Assume that there are $\sigma > 0$, x_0, $x_1 \in X$, such that*

$$\Phi(x_0) = J(x_0) = 0, \qquad \Phi(x_1) > \sigma,$$

Because of a surprising coincidence of names within the same Department, we have to point out that the first author was born on August 4, 1968.

Corresponding author contact: chinni@dipmat.unime.it

Received February 22, 2006

$$\sup_{x\in\Phi^{-1}(]-\infty,\sigma])} J(x) < \sigma\frac{J(x_1)}{\Phi(x_1)}\cdot$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,\sigma])} J(x) < \rho < \sigma\,\frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\rho-J(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\rho-J(x)))\cdot$$

Theorem 1.1 ([7], Theorem 4)**.** *Let X be a reflexive real Banach space, $I\subseteq\mathbb{R}$ an interval, and $\Psi : X\times I\to\mathbb{R}$ a function such that $\Psi(x,\cdot)$ is concave in I for all $x\in X$, $\Psi(\cdot,\lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in X for all $\lambda\in I$. Further, assume that*

$$\sup_{\lambda\in I}\inf_{x\in X}\Psi(x,\lambda) < \inf_{x\in X}\sup_{\lambda\in I}\Psi(x,\lambda)\cdot$$

Then, for each $\alpha>\sup_I\inf_X\Psi$ there exists a non-empty open set $A_\alpha\subseteq I$ with the following property: for every $\lambda\in A_\alpha$ and every sequentially weakly lower semicontinuous functional $H : X\to\mathbb{R}$, there exists $\delta_{\lambda,H}>0$ such that, for each $\mu\in]0,\delta_{\lambda,H}[$, the functional $\Psi(\cdot,\lambda)+\mu H(\cdot)$ has at least two local minima lying in the set $\{x\in X : \Psi(x,\lambda)<\alpha\}$.

Before introducing our results, we precise some notation. On the Sobolev space $W_0^{1,p}(\Omega)$ we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p\,dx\right)^{\frac{1}{p}}\cdot$$

We denote by k the constant

$$k := \sup\left\{\frac{\max_{x\in\overline{\Omega}}|u(x)|}{\|u\|} : u\in W_0^{1,p}(\Omega), u\neq 0\right\}.$$

The weak solutions of $(D_{\lambda,\mu})$ are the functions $u\in W_0^{1,p}$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx = \lambda\int_\Omega f(x,u(x))v(x)dx + \mu\int_\Omega g(x,u(x))v(x)dx$$

for each $v\in W_0^{1,p}$. We put

$$F(x,t) = \int_0^t f(x,\xi)d\xi$$

for each $(x,t)\in\Omega\times\mathbb{R}$.

Fix $x_0\in\Omega$ and $D>0$ such that $B(x_0,D)\subseteq\Omega$ where $B(x_0,D)$ denotes the open ball of $\mathbb{R}^N$ centered on x_0 and having radius D. Moreover, we put

$$m := \left[\frac{\omega}{D^{p-N}}\left(1-\frac{1}{2^N}\right)\right]^{\frac{1}{p}}$$

where $\omega = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ is the measure of unit ball of $\mathbb{R}^N$ and Γ is the Gamma function.

2. MAIN RESULTS

Theorem 2.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $sup_{|\xi|\le s}|f(\cdot,\xi)| \in L^1(\Omega)$ for each $s > 0$. Assume that there exist two positive numbers c, h with $c < 2hmk$ such that*

(i) $F(x,\xi) \ge 0$ *for each* $(x,\xi) \in B(x_0, D) \times [0,h]$;

(ii) $\displaystyle\int_\Omega \sup_{t\in[-c,c]} F(x,t)dx < \left(\frac{c}{2hmk}\right)^p \int_{B(x_0,\frac{D}{2})} F(x,h)dx$;

(iii) $\displaystyle\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^p} \le 0.$

Then, there exist a number $r \in \mathbb{R}$ and an open interval $A \subseteq [0,+\infty[$ with the following property: for every $\lambda \in A$ and for every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ with $sup_{|\xi|\le s}|g(\cdot,\xi)| \in L^1(\Omega)$ for each $s > 0$, there exists $\delta > 0$ such that, for each $\mu \in]0,\delta[$, the problem $(D_{\lambda,\mu})$ has at least two weak solutions whose norms are less than r.

Proof. We put $X = W_0^{1,p}(\Omega)$ and we define the functionals Φ and J as follows

$$\Phi(u) = \frac{1}{p}||u||^p \quad \text{and} \quad J(u) = \int_\Omega F(x,u(x))\,dx$$

for each $u \in X$. Let $\bar{u} \in X$ defined by

$$(1) \qquad \bar{u}(x) = \begin{cases} 0 & x \in \Omega \setminus B(x_0,D) \\ h & x \in B(x_0,\frac{D}{2}) \\ \frac{2h}{D}(D - |x-x_0|) & x \in B(x_0,D) \setminus B(x_0,\frac{D}{2}) \end{cases}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^N$. We have

$$\Phi(\bar{u}) = \frac{1}{p}\int_\Omega |\nabla\bar{u}(x)|^p\,dx = \frac{1}{p}\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \frac{2^p h^p}{D^p}\,dx = \frac{1}{p}(2hm)^p$$

and, by (i),

$$J(\bar{u}) = \int_\Omega F(x,\bar{u}(x))\,dx \ge \int_{B(x_0,\frac{D}{2})} F(x,h)\,dx \cdot$$

Now, taking into account that, for every $u \in X$, one has

$$\max_{x\in\overline{\Omega}} |u(x)| \le k||u||,$$

and put $\sigma = \frac{1}{p}\left(\frac{c}{k}\right)^p$, condition (ii) assures that

$$\sup_{u\in\Phi^{-1}(]-\infty,\sigma])} (J(u)) \le \int_\Omega \sup_{t\in[-c,c]} F(x,t)dx < \sigma\,\frac{J(\bar{u})}{\Phi(\bar{u})}\cdot$$

At this point, chosen

$$\sup_{x\in\Phi^{-1}(]-\infty,\sigma])} J(u) < \rho < \sigma\frac{J(\bar{u})}{\Phi(\bar{u})}),$$

Proposition 1.1 assures that

$$\sup_{\lambda\geq 0}\inf_{u\in X}\Psi(u,\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}\Psi(u,\lambda)$$

where

$$\Psi(u,\lambda) = \Phi(u) - \lambda J(u) + \lambda\rho$$

for each $(u,\lambda) \in X\times[0,+\infty[$. We apply Theorem 1.1 to the functional Φ by choosing $I = [0,+\infty[$.

Easily, we can observe that $\Psi(u,\cdot)$ is concave in I for each $u \in X$ while classical arguments provide the sequential weak lower semicontinuity and the continuity of $\Psi(\cdot,\lambda)$ for each $\lambda \in I$.

Now, we want to prove that the functional $\Psi(\cdot,\lambda)$ is coercive for each $\lambda \in I$. It is obvious that $\Psi(\cdot,0)$ is coercive. Fixed $\lambda \in]0,+\infty[$ and $0 < \epsilon < \dfrac{1}{p\lambda}$, condition (iii) implies that there exists $b_\epsilon \in L^1(\Omega)$ such that

$$F(x,\xi) \leq \epsilon\,|\xi|^p + b_\epsilon(x)$$

for all $x \in X$ and $\xi \in \mathbb{R}$. Then, for each $u \in X$, we have that

$$\Psi(u,\lambda) \geq \left(\frac{1}{p} - \lambda\epsilon\right)\|u\|^p - \lambda\int_\Omega b_\epsilon(x)dx + \lambda\rho$$

i.e. $\Psi(\cdot,\lambda)$ is coercive. Fixed $\alpha > \sup_{\lambda\geq 0}\inf_{u\in X}\Psi(u,\lambda)$, Theorem 1.1 ensures in particular that there exists an open interval $]a,b[\subseteq I$ with the following property: for every $\lambda \in]a,b[$ and every Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ with $sup_{|\xi|\leq s}|g(\cdot,\xi)| \in L^1(\Omega)$ for each $s > 0$, there exists $\delta > 0$ such that, for each $\mu \in]0,\delta[$, the functional $E(u) = \Psi(u,\lambda) - \mu H_g(u)$ has at least two local minima lying in the set $\{u \in X : \Psi(u,\lambda) < \alpha\}$, where H_g is the weakly sequentially lower semicontinuous functional defined by

$$H_g(u) = \int_\Omega\left(\int_0^{u(x)} g(x,\xi)\,d\xi\right)dx$$

for each $u \in X$. These two local minima are critical points of E and then are weak solutions of the problem $(D_{\lambda,\mu})$.

Now we observe that

$$\bigcup_{\lambda\in]a,b[}\{u \in X : \Psi(u,\lambda) < \alpha\} \subseteq \{u \in X : \Psi(u,a) \leq \alpha\} \cup \{u \in X : \Psi(u,b) \leq \alpha\}$$

and so

$$S := \bigcup_{\lambda\in]a,b[}\{u \in X : \Psi(u,\lambda) < \alpha\}$$

is bounded. The conclusion follows taking $A =]a,b[$ and $r = \sup_{u\in S}\|u\|$. □

To obtain three solutions of $(D_{\lambda,\mu})$ instead of two, we add another hypothesis on the function g.

Theorem 2.2. *Let assume the same hypotheses of Theorem 2.1. Then, there exists an open interval $A \subseteq [0, +\infty[$ such that, for every $\lambda \in A$ and every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ with $sup_{|\xi|\leq s}|g(\cdot, \xi)| \in L^1(\Omega)$ for each $s > 0$, and*

(iv) $\limsup_{|\xi|\to+\infty} \dfrac{\sup_{x\in\Omega} \int_0^\xi g(x,t)\,dt}{|\xi|^p} < +\infty,$

there exists $\delta > 0$ such that, for each $\mu \in]0, \delta[$, the problem $(D_{\lambda,\mu})$ has at least three weak solutions.

Proof. Let $A \subseteq [0, +\infty[$ be an open interval as in the conclusion of Theorem 2.1. In particular, fixed a Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ with $sup_{|\xi|\leq s}|g(\cdot, \xi)| \in L^1(\Omega)$ for each $s > 0$ and satisfying (iv), for each $\lambda \in A$, there exists $\delta > 0$ such that for every $\mu \in]0, \delta[$ the problem $(D_{\lambda,\mu})$ has at least two solutions which are critical points of the functional

$$E(u) = \Psi(u, \lambda) - \mu H_g(u).$$

In order to obtain a third solution of $(D_{\lambda,\mu})$, we prove the coercivity of E. From (iii) there exist $a > 0$ and $l \in L^1(\Omega)$ such that

$$\int_0^\xi g(x,t)\,dt \leq a|\xi|^p + l(x)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}$. Then, for each $u \in X$, we have

$$H_g(u) = \int_\Omega \left(\int_0^{u(x)} g(x, \xi)\,d\xi \right) dx \leq a\|u\|^p + \int_\Omega l(x)dx.$$

Fix $\lambda \in A$ and $0 < \bar{\delta} < \min\{\delta, \frac{1}{ap}\}$. Then, for each $\mu \in]0, \bar{\delta}[$, choosen $0 < \epsilon < \frac{1}{\lambda+1}(\frac{1}{p} - \mu a)$, condition (iii) implies that there exists $b_\epsilon \in L^1(\Omega)$ such that the inequality

$$\Psi(u, \lambda) \geq \left(\frac{1}{p} - \lambda\,\epsilon\right)\ \|u\|^p - \lambda \int_\Omega b_\epsilon(x)dx\ +\ \lambda\rho$$

holds for each $u \in X$. Then we have

$$E(u) \geq \left(\frac{1}{p} - \lambda\,\epsilon - \mu\,a\right)\ \|u\|^p - \lambda \int_\Omega b_\epsilon(x)dx + \lambda\rho - \mu \int_\Omega l(x)dx$$

for each $u \in X$. The last condition provides the coercivity of E.

Standard arguments assure that the functional Φ' admits a continuous inverse on X^* while J' and H'_g are compact. Then, by Example 38.25 of [8] we deduce that the functional E has the Palais-Smale property. Finally, by using Corollary 1 of [4] and taking into account that the functional E is C^1 on X, there exists a third critical point of E which is a third solution of problem $(D_{\lambda,\mu})$. □

Now, we present an example in which Theorem 2.2 is applied.

Example 2.1. Let $N = 3$, $p = 4$ and $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$. In this case $k = (\frac{9}{4\pi})^{\frac{1}{4}}$. Choose $x_0 = 0$ and $D = 1$. Hence $m = (\frac{7\pi}{6})^{\frac{1}{4}}$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ and $a : \Omega \to \mathbb{R}$ be defined by setting

$$\alpha(\xi) = \begin{cases} \xi^{2\beta} & \text{if } \xi \le 1 \\ \xi^{\alpha} & \text{if } \xi > 1 \end{cases}$$

for each $\xi \in \mathbb{R}$, with $\beta > \frac{3}{2}$ and $0 < \alpha < 3$ and

$$a(x) = \begin{cases} 1 & \text{if } x \in B(0, \frac{1}{2}) \\ 2(1 - |x|) & \text{if } x \in B(0,1) \setminus B(0, \frac{1}{2}) \end{cases}$$

for each $x \in \Omega$.

Then, there exists an open interval $A \subseteq [0, +\infty[$ with the following property: for every $\lambda \in A$, for every continuous function $b : \Omega \to \mathbb{R}$ and every $\gamma \in]0, 3]$, there exists $\delta > 0$ such that , for each $\mu \in]0, \delta[$, the problem

$$(D) \qquad \begin{cases} -\Delta_4 u = \lambda a(x)\alpha(u) + \mu b(x)|u|^{\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least two nontrivial weak solutions.

Let $f(x,t) = a(x)\alpha(t)$ and $c, h \in]0, 1[$ such that $(\frac{15}{4}(2mk)^4)^{\frac{1}{3-2\beta}} < \frac{c}{h} < 1$. With such choices one has

$$F(x,\xi) = \int_0^{\xi} a(x)\alpha(t)dt = \begin{cases} a(x)\dfrac{\xi^{2\beta+1}}{2\beta+1} & \text{if } \xi \le 1 \\ a(x)\left(\dfrac{1}{2\beta+1} + \dfrac{1}{\alpha+1}\left(\xi^{\alpha+1} + 1\right)\right) & \text{if } \xi > 1 \end{cases}$$

for each $(x,\xi) \in \Omega \times \mathbb{R}$ and so conditions (i) and (iii) follows obviously. On the other hand it results that

$$\int_{B(0,1)} \sup_{t \in [-c,c]} F(x,t)\, dx = \frac{c^{2\beta+1}}{2\beta+1} \int_{B(0,1)} a(x)\, dx = \frac{5}{8}\pi \, \frac{c^{2\beta+1}}{2\beta+1}$$

and condition (ii) follows thaking into account that

$$\frac{1}{(2mk)^4}\left(\frac{c}{h}\right)^4 \int_{B(0,\frac{1}{2})} F(x,h)\, dx = \frac{1}{(2mk)^4}\left(\frac{c}{h}\right)^4 \frac{h^{2\beta+1}}{2\beta+1} \int_{B(0,\frac{1}{2})} a(x)\, dx =$$

$$\frac{1}{(2mk)^4}\left(\frac{c}{h}\right)^4 \frac{h^{2\beta+1}}{2\beta+1}\,\frac{\pi}{6} = \frac{1}{(2mk)^4}\left(\frac{c}{h}\right)^{3-2\beta} \frac{c^{2\beta+1}}{2\beta+1}\,\frac{\pi}{6} > \frac{5}{8}\pi\,\frac{c^{2\beta+1}}{2\beta+1}.$$

Finally, condition (iv) is satisfied with $g(x,t) = b(x)|t|^{\gamma}$for each $(t,x) \in \Omega \times \mathbb{R}$.

Applying Theorem 2.1 instead of Theorem 2.2, with f defined as in Example 2.1, we obtain the following conclusion: there exist a number $r \in \mathbb{R}$ and an open interval $A \subseteq [0, +\infty[$ with the following property: for every $\lambda \in A$, for every continuous function $b : \Omega \to \mathbb{R}$ and every $\gamma > 0$, there exists $\delta > 0$ such that , for each $\mu \in]0, \delta[$, the problem (D) has at least a non trivial weak solution whose norm is less than r .

REFERENCES

[1] G. BONANNO and N. GIOVANNELLI. *An eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities.* J. Math. Anal. Appl., **13**:596–604, 2005.

[2] G. BONANNO and R. LIVREA. *Multiplicity theorems for the Dirichlet problem involving the p-Laplacian.* Nonlinear Analysis, **54**:1–7, 2003.

[3] F. CAMMAROTO, A. CHINNÌ and DI BELLA B. *Multiple solutions for a two points boundary value problem.* J. Math. Anal. Appl., to appear.

[4] P. PUCCI and J. SERRIN. *A mountain pass theorem.* J. Differential Equations, **60**:142–149, 1985.

[5] B. RICCERI. *Existence of three solutions for a class of elliptic eigenvalue problems.* Math. Comput. Modelling, **32**:1485–1494, 2000.

[6] B. RICCERI. *On a three critical points theorem.* Arch. Math., **75**:220–226, 2000.

[7] B. RICCERI. *Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set.* Topology Appl., to appear.

[8] E. ZEIDLER. *Nonlinear Functional Analysis and Applications, Vol. III.* Springer, 1985.