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SEMILINEAR EVOLUTION EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT. We are concerned with the study of semilinear evolution equations with nonlocal initial conditions. We provide sufficient conditions on the nonlinearity which allow the use of variants of the nonlinear alternative to prove the existence of at least one solution. Our second result presents a novel growth condition splitted into two parts, one for the subinterval containing the points involved by the initial conditions, and another for the rest of the interval.

Keywords. Evolution equation, Nonlocal initial condition, Mild solution, Fixed point.

AMS (MOS) Subject Classification. 34G20, 47D06, 47H10, 47N20.

1. INTRODUCTION

In this paper we are concerned with the existence of mild solutions of a nonlocal Cauchy problem for a semilinear evolution equation. In fact, we consider the following Cauchy problem with nonlocal initial conditions

$$u'(t) + Au(t) = f(t, u(t)) \qquad 0 < t < 1 \tag{1.1}$$

$$u(0) + \sum_{k=1}^{m} a_k u(t_k) = 0 \tag{1.2}$$

where $-A$ is the infinitesimal generator of a compact C_0 semigroup $\{T(t)\}_{t\geq 0}$ of operators on a real Banach space E, f is a given function and a_k are real numbers and t_k, $k = 1, 2, \ldots, m$ are given points with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m < 1$. Nonlocal Cauchy problems have attracted the attention of many researchers (see for instance [1–4, 6, 8, 9] and the references therein). We provide sufficient conditions on the nonlinearity f and the numbers a_k in order to obtain a priori bounds on solutions of a one-parameter family of problems related to the original. Our assumptions are less restrictive than those imposed in earlier works. We do not assume that f maps bounded sets into precompact sets. Also, if we let $g(u) = \sum_{k=1}^{m} a_k u(t_k)$, then our

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problem is similar to those considered by [4] and [9], but we can easily see that g is not uniformly bounded. Our second result presents a novel growth condition splitted in two parts, one for the subinterval containing the points involved by the initial conditions, and another for the rest of the interval.

The importance of nonlocal conditions in various disciplines is discussed in [4–7] and the references therein.

2. PRELIMINARIES

Let E be a real Banach space with norm $\|\cdot\|$, $BL(E)$ is the Banach space of bounded linear operators on E with norm $\|\cdot\|_{op}$; $-A$ is the infinitesimal generator of a compact C_0 semigroup $\{T(t)\}_{t\geq 0}$ (i.e. $T(t)$ is a compact operator for each $t>0$) on E; J is the real interval $[0,1]$. Let $X := C(J;E)$. For $u \in X$ define its norm by $|u|_\infty := \sup\{\|u(t)\|;\ \ t\in J\}$. Let $M := \sup\{\|T(t)\|_{op};\ t \in J\}$. The following results play an important role in our main results.

Theorem 2.1. (Schaefer [11]) *Let Y be a normed space, Φ a continuous mapping of Y into Y which is compact on each bounded subset of Y. Then either*

(i) *the equation $x = \lambda\Phi x$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

Theorem 2.2. (O'Regan [10]) *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$, $G(\overline{U})$ is bounded and $G : \overline{U} \to C$ is given by $G = G_1 + G_2$ where $G_1 : \overline{U} \to E$ is completely continuous, and $G_2 : \overline{U} \to E$ is a nonlinear contraction (i.e. there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|G_2(x) - G_2(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in \overline{U}$). Then either,*

(A1) *G has a fixed point in $\overline{U}$, or*
(A2) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda G(u)$.*

3. EXISTENCE OF MILD SOLUTIONS

In order to study our problem, we shall assume that the operator $T(t)$ is compact for each $t > 0$, and that there exists a bounded operator B on $D(B) = E$ given by the formula

$$B := \left[I + \sum_{k=1}^{m} a_k T(t_k) \right]^{-1}.$$

This is possible, for instance if $\sum_{k=1}^{m} |a_k| < 1/M$.

Our first result is based on the following assumption

(H1) $f : J \times E \to E$ is continuous and

$$\|f(t,u)\| \le h(t)\phi(\|u\|)$$

for all $t \in J$, $u \in E$, where $h \in L^1(J;[0,\infty))$ and $\phi : [0,+\infty) \to (0,+\infty)$ is continuous nondecreasing such that

$$\limsup_{R\to+\infty} \frac{R}{\left(M^2\|B\|_{op}\sum_{k=1}^{m}|a_k|\int_0^{t_k} h(s)ds + M\|h\|_{L^1}\right)\phi(R)} > 1.$$

Definition 3.1. A mild solution of (1.1), (1.2) is a continuous solution of the integral equation

$$u(t) = -\sum_{k=1}^{m} a_k T(t) B\int_0^{t_k} T(t_k - s)f(s,u(s))ds + \int_0^t T(t-s)f(s,u(s))ds.$$

Theorem 3.2. *Suppose that the assumption* (H1) *is satisfied. Then the nonlocal Cauchy problem* (1.1), (1.2) *has at least one mild solution.*

Proof. Consider a one-parameter family of problems

$$\begin{cases} u'(t) + Au(t) = \lambda f(t,u(t)), & t \in J \\ u(0) + \sum_{k=1}^{m} a_k u(t_k) = 0 \end{cases} \tag{3.1}$$

where $0 \le \lambda \le 1$. Define $\Phi : X \to X$ by the formula

$$(\Phi u)(t) := -\sum_{k=1}^{m} a_k T(t) B\int_0^{t_k} T(t_k - s)f(s,u(s))ds + \int_0^t T(t-s)f(s,u(s))ds.$$

A mild solution of (3.1) is a solution of the abstract equation

$$u = \lambda\Phi u \tag{3.2}$$

and conversely.

Step 1. The solutions of (3.2) are a priori bounded. For, let u be any solution of (3.2) and let $R_0 := |u|_\infty$. It follows from the integral equation that for all $t > 0$

$$\begin{aligned} \|u(t)\| \le\ & \sum_{k=1}^{m}|a_k|\,\|T(t)\|_{op}\|B\|_{op}\int_0^{t_k}\|T(t_k-s)\|_{op}\,\|f(s,u(s))\|\,ds \\ & + \int_0^t \|T(t-s)\|_{op}\,\|f(s,u(s))\|\,ds. \end{aligned}$$

Condition (H1) implies that for all $t > 0$

$$\begin{aligned} \|u(t)\| \le\ & \sum_{k=1}^{m}|a_k|\,\|T(t)\|_{op}\|B\|_{op}\int_0^{t_k}\|T(t_k-s)\|_{op}\,h(s)\phi(\|u(s)\|)ds \\ & + \int_0^t \|T(t-s)\|_{op}\,h(s)\phi(\|u(s)\|)ds. \end{aligned}$$

Since ϕ is nondecreasing we have that for all $t > 0$

$$\begin{aligned}\|u(t)\| \ &\leq \ \sum_{k=1}^{m} |a_k| \, \|T(t)\|_{op} \|B\|_{op} \int_0^{t_k} \|T(t_k - s)\|_{op} \, h(s)\phi(R_0)ds \\ &+ \ \int_0^t \|T(t-s)\|_{op} \, h(s)\phi(R_0)ds.\end{aligned}$$

Hence

$$\|u(t)\| \leq \left(M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| \int_0^{t_k} h(s)ds + M\|h\|_{L^1} \right) \phi(R_0) \quad \text{for all } t > 0.$$

This implies that

$$\frac{R_0}{\left(M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| \int_0^{t_k} h(s)ds + M\|h\|_{L^1} \right) \phi(R_0)} \leq 1. \tag{3.3}$$

Now the condition on ϕ implies that there exists $R^* > 0$ such that for all $R > R^*$ we have

$$\frac{R}{\left(M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| \int_0^{t_k} h(s)ds + M\|h\|_{L^1} \right) \phi(R)} > 1. \tag{3.4}$$

Comparing inequalities (3.3) and (3.4) we see that

$$R_0 \leq R^*.$$

Thus, we have obtained that any solution u of (3.2) satisfies

$$|u|_\infty \leq R^* \quad \text{i.e.} \quad \max_{t \in J} \|u(t)\| \leq R^*.$$

Therefore, all possible solutions of (3.2) are a priori bounded independently of λ. In fact, these solutions are in the closed convex subset

$$S_{R^*} := \{u \in X; |u|_\infty \leq R^*\}.$$

Also, the continuity of Φ follows from the continuity of f.

Step 2. Φ is completely continuous. In order to prove this we must show that $\Phi(S_{R^*})$ is a uniformly equicontinuous family of functions and the set $S_{R^*}(t) := \{(\Phi u)(t); \ u \in S_{R^*}\}$ is precompact in E, for every $t \in J$. Let $\sigma_1, \sigma_2 \in J$ with $\sigma_1 < \sigma_2$. Then

$$\begin{aligned}&\|(\Phi u)(\sigma_1) - (\Phi u)(\sigma_2)\| \leq \\ &\|T(\sigma_1) - T(\sigma_2)\| \, \|B\|_{op} \sum_{k=1}^{m} |a_k| \int_0^{t_k} \|T(t_k - s)\|_{op} \, \|f(s, u(s))\|ds \\ &+ \int_0^{\sigma_1} \|T(\sigma_1) - T(\sigma_2 - s)\|_{op} \, \|f(s, u(s))\|ds \\ &+ \int_{\sigma_1}^{\sigma_2} \|T(\sigma_2 - s)\|_{op} \, \|f(s, u(s))\|ds.\end{aligned}$$

Since $f \in C(J \times E; E)$ it follows that there exists $\rho > 0$ such that

$$\|f(s, u(s))\| \leq \rho$$

for all $s \in J$ and $u \in S_{R^*}$. Let

$$\gamma := \|B\|_{op} \rho \sum_{k=1}^{m} |a_k| \int_0^{t_k} \|T(t_k - s)\|_{op} ds.$$

It follows from the above that

$$\begin{aligned}
\|(\Phi u)(\sigma_1) - (\Phi u)(\sigma_2)\| \leq \;& \gamma \|T(\sigma_1) - T(\sigma_2)\|_{op} \\
& + \rho \int_0^{\sigma_1} \|T(\sigma_1 - s) - T(\sigma_2 - s)\|_{op} ds \\
& + \rho \int_{\sigma_1}^{\sigma_2} \|T(\sigma_2 - s)\|_{op} ds.
\end{aligned}$$

The right-hand side does not depend on $u \in S_{R^*}$ and tends to zero when σ_2 tends to σ_1 because of the continuity of $T(t)$ in the uniform operator topology for $t > 0$, which follows from the compactness of $T(t)$ for $t > 0$. Hence $\Phi(S_{R^*})$ is an equicontinuous family of functions.

Next, consider the set $S_{R^*}(t) := \{(\Phi u)(t); u \in S_{R^*}\}$, $t \in J$. Let $t > 0$ and $0 < \epsilon < t$. For $u \in S_{R^*}$ define

$$\begin{aligned}
(\Phi_\epsilon u)(t) := \;& - \sum_{k=1}^{m} a_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s)) ds \\
& + \int_0^{t-\epsilon} T(t - s) f(s, u(s)) ds \\
= \;& - \sum_{k=1}^{m} a_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s)) ds \\
& + T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon) f(s, u(s)) ds.
\end{aligned}$$

Since $T(t)$ is compact for every $t > 0$, the set $S_{R^*,\epsilon}(t) := \{(\Phi_\epsilon u)(t); u \in S_{R^*}\}$ is precompact in E, for every $\epsilon \in (0, t)$. Moreover, for every $u \in S_{R^*}$ we have

$$\begin{aligned}
\|(\Phi_\epsilon u)(t) - (\Phi u)(t)\| & \leq \int_{t-\epsilon}^{t} \|T(t - s) f(s, u)(s))\| ds \\
& \leq M \rho \epsilon.
\end{aligned}$$

This shows that the set $S_{R^*}(t) := \{(\Phi u)(t); u \in S_{R^*}\}$ is precompact in E. It follows from the theorem of Ascoli-Arzela that $\Phi(S_{R^*})$ is a precompact subset of X.

Notice that $S(\Phi) := \{u \in X; u = \lambda \Phi u, 0 < \lambda < 1\}$ is bounded as we proved in the first step. Therefore by Theorem 2.1, the operator Φ has a fixed point in S_{R^*}, and any fixed point of Φ is a mild solution of (1.1), (1.2). This completes the proof of our first result. □

For our second result, we shall assume that the following conditions hold.

(H2) There exists $M_0 > 0$ and $\omega \in C(J \times [0, +\infty); (0, +\infty))$ nondecreasing with respect to its second variable, with the property that

$$\frac{1}{R}\int_0^{t_m} \omega(t, R)\, dt < \frac{1}{M(1 + M\|B\|_{op}\sum_{k=1}^{m}|a_k|)}$$

for all $R > M_0$ and

$$\|f(t, u)\| \leq \omega(t, \|u\|)$$

for all $t \in [0, t_m]$ and $u \in E$.

(H3) There exists $q \in L^1([t_m, 1]; \mathbb{R}_+)$ with $M\int_{t_m}^1 q(s)ds \leq 1$, and $\Gamma \in C(\mathbb{R}_+; \mathbb{R}_+)$ nondecreasing with $\Gamma(z) < z$ for $z > 0$, such that

$$\|f(t, u_1) - f(t, u_2)\| \leq q(t)\Gamma(\|u_1 - u_2\|)$$

for all $t \in [t_m, 1]$ and $u_1, u_2 \in E$.

Theorem 3.3. *Assume that* (H2) *and* (H3) *are satisfied. Then the nonlocal problem* (1.1), (1.2) *has at least one mild solution.*

Proof. Consider the one-parameter family of problems (3.1) and the equivalent integral equation (3.2). Write

$$\Phi(u) = G_1(u) + G_2(u)$$

where

$$G_1(u)(t) = \begin{cases} -\sum_{k=1}^{m} a_k T(t)B\int_0^{t_k} T(t_k - s)f(s, u(s))ds + \int_0^{t} T(t - s)f(s, u(s))ds \\ \qquad\qquad \text{for } \ t < t_m \\ -\sum_{k=1}^{m} a_k T(t)B\int_0^{t_k} T(t_k - s)f(s, u(s))ds + \int_0^{t_m} T(t - s)f(s, u(s))ds \\ \qquad\qquad \text{for } \ t \geq t_m \end{cases}$$

and

$$G_2(u)(t) = \begin{cases} 0, & \text{for } \ t < t_m \\ \int_{t_m}^{t} T(t - s)f(s, u(s))ds, & \text{for } \ t \geq t_m. \end{cases}$$

We want to show that there exists $\delta > 0$ such that any possible solution u of (3.2) satisfies $|u|_\infty \leq \delta$.

For $t \in [0, t_m]$ we have, for $0 \leq \lambda \leq 1$

$$\begin{aligned}
\|u(t)\| &= \lambda \left\| -\sum_{k=1}^{m} a_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s)) ds + \int_0^t T(t-s) f(s,u(s)) ds \right\| \\
&\leq \sum_{k=1}^{m} |a_k| \, \|T(t)\|_{op} \, \|B\|_{op} \int_0^{t_k} \|T(t_k - s)\|_{op} \, \|f(s,u(s))\| \, ds \\
&+ \int_0^t \|T(t-s)\|_{op} \, \|f(s,u(s))\| \, ds \\
&\leq (M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| + M) \int_0^{t_m} \|f(s,u(s))\| \, ds \\
&\leq M(1 + M \|B\|_{op} \sum_{k=1}^{m} |a_k|) \int_0^{t_m} \omega(s, \|u(s)\|) ds.
\end{aligned}$$

Letting $\beta = \max\{\|u(t)\| \, ; 0 \leq t \leq t_m\}$ we see from the above inequality that (recall ω is nondecreasing in its second variable)

$$\frac{1}{\beta} \int_0^{t_m} \omega(s, \beta) ds \geq \frac{1}{M(1 + M \|B\|_{op} \sum_{k=1}^{m} |a_k|)}.$$

Now the condition on ω implies that $\beta \leq M_0$, i.e.

$$\max\{\|u(t)\| \, ; 0 \leq t \leq t_m\} \leq M_0.$$

Next, let $t \in [t_m, 1]$. Then

$$\begin{aligned}
\|u(t)\| &= \lambda \left\| -\sum_{k=1}^{m} a_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s)) ds + \int_0^t T(t-s) f(s,u(s)) ds \right\| \\
&\leq (M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| + M) \int_0^{t_m} \|f(s,u(s))\| \, ds + M \int_{t_m}^t \|f(s,u(s))\| \, ds \\
&\leq (M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| + M) \int_0^{t_m} \omega(s, \|u(s)\|) ds + M \int_{t_m}^t q(s) \Gamma\left(\|u(s)\|\right) ds \\
&\leq (M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| + M) \int_0^{t_m} \omega(s, M_0) ds + M \int_{t_m}^t q(s) \Gamma\left(\|u(s)\|\right) ds.
\end{aligned}$$

Let

$$V(t) := (M^2 \|B\|_{op} \sum_{k=1}^{m} |a_k| + M) \int_0^{t_m} \omega(s, M_0) ds + M \int_{t_m}^t q(s) \Gamma\left(\|u(s)\|\right) ds.$$

Then, for all $t \in [t_m, 1]$

$$\|u(t)\| \leq V(t) \quad \text{and} \quad V'(t) = M q(t) \Gamma\left(\|u(t)\|\right).$$

Since Γ is nondecreasing we have

$$V'(t) \leq M q(t) \Gamma\left(\|V(t)\|\right) \quad \text{for all} \quad t \in [t_m, 1].$$

So that

$$\frac{V'(t)}{\Gamma\left(\|V(t)\|\right)} \leq Mq(t) \quad \text{for all} \;\; t \in [t_m, 1].$$

Hence

$$\int_{t_m}^{t} \frac{V'(s)}{\Gamma\left(\|V(s)\|\right)} ds \leq M \int_{t_m}^{t} q(s)ds \leq M \int_{t_m}^{1} q(s)ds \leq 1.$$

By an obvious change of variables we obtain

$$\int_{V(t_m)}^{V(t)} \frac{dz}{\Gamma(z)} \leq 1.$$

Now we use the fact that $\Gamma\left(z\right) < z,$ to see that $\dfrac{1}{z} < \dfrac{1}{\Gamma\left(z\right)}$. Consequently

$$\int_{V(t_m)}^{V(t)} \frac{dz}{z} \leq 1.$$

Hence

$$\ln \frac{V(t)}{V(t_m)} \leq 1 \quad \text{for all} \;\; t \in [t_m, 1].$$

Thus

$$V(t) \leq eV(t_m) = e(M^2 \left\|B\right\|_{op} \sum_{k=1}^{m} |a_k| + M) \int_0^{t_m} \omega(s, M_0)ds := M_1.$$

It follows that

$$\max\{\|u(t)\|\,; t_m \leq t \leq 1\} \leq M_1.$$

Set $\delta := \max(M_0, M_1)$. Then, we have obtained that any possible solution of (3.2) satisfies $|u|_\infty \leq \delta$.

Consider the set $\Omega := \{u \in X; |u|_\infty < \delta + 1\}$. Recall that mild solutions of Problem (1.1), (1.2) are fixed points of the operator $\Phi : X \to X$, where

$$\Phi u = G_1(u) + G_2(u).$$

We proceed as in the proof of Theorem 3.2 to show that the operator $G_1 : \overline{\Omega} \to X$ is compact. Next, we show that $G_2 : \overline{\Omega} \to X$ is a nonlinear contraction. Indeed, for any $x, y \in \overline{\Omega}$ and $t \in J$ we have from the definition of G_2,

$$\begin{aligned}
\|G_2(x)(t) - G_2(y)(t)\| &= \left\| \int_{t_m}^{t} T(t-s)[f(s,x(s)) - f(s,y(s))]ds \right\| \\
&\leq M \int_{t_m}^{t} \|f(s,x(s)) - f(s,y(s))\|\, ds \\
&\leq M \int_{t_m}^{t} q(s)\Gamma\left(\|x(s) - y(s)\|\right) ds \\
&\leq M\Gamma\left(|x-y|_\infty\right) \int_{t_m}^{t} q(s)ds \\
&\leq \Gamma\left(|x-y|_\infty\right) M \int_{t_m}^{1} q(s)ds.
\end{aligned}$$

Since $M \int_{t_m}^{1} q(s)ds \leq 1$, we deduce that

$$|G_2(x) - G_2(y)|_\infty \leq \Gamma\left(|x-y|_\infty\right).$$

Hence the operator $G_2 : \overline{\Omega} \to X$ is a nonlinear contraction.

Condition (A2) of Theorem 2.2 does not hold because we have seen that solutions u of (3.2) are such that $|u|_\infty \leq \delta$, for all $\lambda \in [0,1]$. So that there is no solution of (3.2) with $u \in \partial\Omega$.

Therefore, Theorem 2.2 implies that Φ has a fixed point in Ω. This shows that Problem (1.1), (1.2) has a solution. This completes the proof of Theorem 3.3. □

Remark 3.4. Theorem 3.2 also follows from Theorem 1.1, even if (H3) is replaced by the following more general condition (H3'):

(H3') There exists $q \in L^1([t_m, 1]; \mathbb{R}_+)$ and $\Gamma \in C\left(\mathbb{R}_+; \mathbb{R}_+\right)$ nondecreasing with

$$\int_{t_m}^{1} q(s)ds < \int_{M_0^*}^{+\infty} \frac{dz}{\Gamma(z)}$$

where

$$M_0^* = (1 + M\left\|B\right\|_{op} \sum_{k=1}^{m} |a_k|) \int_0^{t_m} \omega(s, M_0)ds,$$

such that

$$\|f(t,u)\| \leq q(t)\Gamma\left(\|u\|\right)$$

for all $t \in [t_m, 1]$ and $u \in E$.

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