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NEW OSCILLATIONS CRITERIA OF DELAY PARTIAL DIFFERENCE EQUATIONS

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ABSTRACT. Consider the delay partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) A_{m-k_i,n-l_i} = 0,\ m,n \in N_0,$$

where $\liminf_{m,n\to\infty} P_i(m,n) = p_i \in [0,\infty),\ k_i, l_i \in N_1,\ i = 1,2,\dots,u$. Sufficient conditions for the oscillation of all solutions of the above equation are established in the case when that the corresponding "limiting" equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_i A_{m-k_i,n-l_i} = 0,\ m,n \in N_0,$$

admits non-oscillatory solutions. Oscillation criteria for nonlinear partial difference equation are also derived as applications.

Keywords. Oscillation, Partial difference equations, Delay.

1. INTRODUCTION

Partial difference equations occur frequently in the approximation of solutions of partial differential equations by finite difference methods and some science problems [1–5]. We consider the partial difference equation with several delays

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) A_{m-k_i,n-l_i} = 0,\ m,n \in N_0, \tag{1}$$

where $\{P_i(m,n)\}_{n=1}^{\infty}$ is a double real sequence with $P_i(m,n) \geq 0$ for all large m,n, $k_i, l_i \in N_1, (i = 1,2,\dots,u), N_t = \{t, t+1, t+2, \dots\}$, and

$$P_i(m,n) \geq p_i \in [0,\infty),\ \liminf_{m,n\to\infty} P_i(m,n) = p_i,\ i = 1,2,\dots,u\ . \tag{2}$$

Then, the corresponding limiting equation of (1) is

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_i A_{m-k_i,n-l_i} = 0,\ m,n \in N_0\ , \tag{3}$$

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with the characteristic equation

$$\lambda + \mu - 1 + \sum_{i=1}^{u} p_i \lambda^{-k_i} \mu^{-l_i} = 0. \tag{4}$$

It is well known (for example, see [6]) that all solutions of Eq. (3) oscillate if and only if (4) has no positive roots. In [7], Zhang and Yu showed that all solutions of Eq. (1) oscillate if all solutions of Eq. (3) oscillate. However, the following situation is also possible: all solutions of Eq. (1) oscillate in spite of the fact that the corresponding limiting equation (3) admits non-oscillatory solutions.

Oscillatory properties of equation (1) have been investigated by many authors; see the survey paper [8].

In this paper, we introduce some new techniques to established sufficient conditions for the oscillation of all solutions of Eq. (1) in the case when that the corresponding limiting equation (3) admits non-oscillatory solutions. It is to be pointed out that there is no result on this problem up to now. As applications, we also obtain oscillation criteria for nonlinear partial difference equation with several delays.

By a solution of Eq. (1), we mean a sequence $\{A_{m,n}\}$ which is defined for $m \geq -k^*$, $n \geq -l^*$, where $k^* = \max_{i\geq 1}\{k_i\}$, $l^* = \max_{i\geq 1}\{l_i\}$, and which satisfies (1) for $m, n \in N_0$. A solution $\{A_{m,n}\}$ of (1) is said to be oscillatory if the terms $A_{m,n}$ of the sequence $\{A_{m,n}\}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. MAIN RESULTS

First, we define a sequence $\{\lambda_l\}_{l=1}^{\infty}$ by

$$\lambda_1 = 1,\ \lambda_{l+1} = 1 - \sum_{i=1}^{u} p_i \lambda_l^{-k_i - l_i},\ l = 1, 2, \dots , \tag{5}$$

where $p_i \geq 0$, $i = 1, 2, \dots, u$.

The following lemma will be used to prove our main results.

Lemma 1. *Assume that the sequence $\{\lambda_l\}$ is defined by (5). Then, $\lambda_* \leq \lambda_l \leq 1$ and $\lim_{l\to\infty} \lambda_l = \lambda_*$, where λ_* is the largest root of the equation*

$$\lambda = 1 - \sum_{i=1}^{u} p_i \lambda^{-k_i - l_i} \tag{6}$$

on $(0, 1]$.

The proof is simple and we omit it here.

In the following, we consider linear partial difference inequalities of the form

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) A_{m-k_i, n-l_i} \leq 0,\ m, n \in N_0, \tag{7}$$

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) A_{m-k_i,n-l_i} \geq 0,\ m,n \in N_0. \tag{8}$$

Assume that $p_i, i = 1, 2, \ldots, u$ are sufficiently small such that the equation

$$2\lambda - 1 + \sum_{i=1}^{u} p_i \lambda^{-k_i - l_i} = 0 \tag{9}$$

has positive roots on $(0, 1/2)$. Hence (4) has positive roots, which implies that (3) has nonoscillatory solutions. We will give sufficient conditions for the oscillation of (1) in this case.

Theorem 1. *Assume that* (2) *holds. Further assume that*

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} (\lambda_*^{-k_i-l_i} P_i(m,n) + \lambda_*^{1-k_i-l_i}(P_i(m+1,n) + P_i(m,n+1))) > 1, \tag{10}$$

where λ_* *is the largest root of* (6) *on* $(0,1]$. *Then:*

(i) (7) *has no eventually positive solutions;*
(ii) (8) *has no eventually negative solutions; and*
(iii) *every solution of Eq.* (1) *oscillates.*

Proof. It is easy to prove (ii) and (iii) similarly to the proof of (i). Assume, for the sake a contradiction, that $\{A_{m,n}\}$ is an eventually positive solution of (1). Then, there exist m_1 and n_1 such that $A_{m,n} > 0$ and $A_{m-k_i,n-l_i} > 0$, $i = 1, 2, \ldots, u$, for $m \geq m_1$, $n \geq n_1$. Therefore, from (1), we have

$$A_{m+1,n} < A_{m,n} \quad \text{and } A_{m,n+1} < A_{m,n}, \quad \text{for } m \geq m_1,\ n \geq n_1,$$

which gives

$$A_{m-k_i,n} \geq \lambda_1^{-k_i} A_{m,n} \quad \text{for } m \geq m_1 + k_i,\ n \geq n_1.$$

Hence, we have

$$A_{m-k_i,n-l_i} \geq \lambda_1^{-k_i-l_i} A_{m,n}, \quad \text{for } \ m \geq m_1 + k_i, n \geq n_1 + l_i. \tag{11}$$

Using now (11) and (1) we have

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} p_i \lambda_1^{-k_i-l_i} A_{m,n} \leq 0.$$

Hence, we have

$$A_{m+1,n} \leq A_{m,n}(1 - \sum_{i=1}^{u} p_i \lambda_1^{-k_i-l_i}) = \lambda_2 A_{m,n} \ \text{ for } m \geq m_1 + k_i, n \geq n_1 + l_i,$$

and

$$A_{m,n+1} \leq A_{m,n}(1 - \sum_{i=1}^{u} p_i \lambda_1^{-k_i-l_i}) = \lambda_2 A_{m,n} \quad \text{for } m \geq m_1 + k_i,\ n \geq n_1 + l_i.$$

Hence,

$$A_{m-k_i,n-l_i} \geq \lambda_2^{-k_i-l_i} A_{m,n} \quad \text{for } m \geq m_1 + 2k_i,\ n \geq n_1 + 2l_i.$$

Repeating the above procedure, we get

$$(12)\quad A_{m+1,n} \leq A_{m,n}(1-\sum_{i=1}^{u} p_i \lambda_{l-1}^{-k_i-l_i}) = \lambda_l A_{m,n} \quad \text{for } m \geq m_1+k_i,\ n \geq n_1+(l-1)l_i,$$

and

$$(13)\quad A_{m,n+1} \leq A_{m,n}(1-\sum_{i=1}^{u} p_i \lambda_{l-1}^{-k_i-l_i}) = \lambda_l A_{m,n} \quad \text{for } m \geq m_1+k_i,\ n \geq n_1+(l-1)l_i.$$

Hence,

$$(14)\qquad A_{m-k_i,n-l_i} \geq \lambda_l^{-k_i-l_i} A_{m,n}, \quad \text{for } m \geq m_1 + lk_i,\ n \geq n_1 + ll_i,$$

where

$$\lambda_l = 1 - \sum_{i=1}^{u} p_i \lambda_{l-1}^{-k_i-l_i}.$$

Since $\lim_{l\to\infty} \lambda_l = \lambda_*$, for a sequence $\{\varepsilon_l\}$ with $\varepsilon_l > 0$ and $\varepsilon_l \to 0$ as $l \to \infty$, by (12), (13), and (14) there exists a double sequence $\{m_l, n_l\}$ such that $m_l, n_l \to \infty$ as $l \to \infty$ and

$$(15)\qquad A_{m+1,n} \leq (\lambda_* + \varepsilon_l) A_{m,n}, \quad \text{for } m \geq m_l,\ n \geq n_l,$$

$$(16)\qquad A_{m,n+1} \leq (\lambda_* + \varepsilon_l) A_{m,n}, \quad \text{for } m \geq m_l,\ n \geq n_l,$$

and

$$(17)\qquad A_{m-k_i,n-l_i} \geq (\lambda_* + \varepsilon_l)^{-k_i-l_i} A_{m,n} \quad \text{for } m \geq m_l + k_i,\ n \geq n_1 + l_i.$$

From (1) and (17), we have

$$(18)\qquad A_{m,n} \geq \sum_{i=1}^{u} P_i(m,n)(\lambda_* + \varepsilon_1)^{1-k_i-l_i} A_{m-1,n}$$

and

$$(19)\qquad A_{m,n} \geq \sum_{i=1}^{u} P_i(m,n)(\lambda_* + \varepsilon_1)^{1-k_i-l_i} A_{m,n-1}.$$

Dividing (1) by $A_{m,n}$, we have

$$(20)\quad 1 = \frac{A_{m+1,n} + A_{m,n+1}}{A_{m,n}} + \sum_{i=1}^{u} P_i(m,n) \frac{A_{m-k_i,n-l_i}}{A_{m,n}} \quad \text{for } m \geq m_l + k^*,\ n \geq n_l + l^*.$$

From (17)–(20), we have

$$1 \geq \sum_{i=1}^{u} ((\lambda_* + \varepsilon_l)^{-k_i-l_i} P_i(m,n) + (\lambda_* + \varepsilon_l)^{1-k_i-l_i} (P_i(m+1,n) + P_i(m,n+1))).$$

Letting $l \to \infty$, the above inequality implies

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} (\lambda_*^{-k_i-l_i} P_i(m,n) + \lambda_*^{1-k_i-l_i}(P_i(m+1,n) + P_i(m,n+1))) \le 1,$$

which contradicts (10) and completes the proof. □

Theorem 2. *Assume that* (2) *holds, and* (10) *is replaced by*

$$\limsup_{m,n\to\infty} \Big(\sum_{i=1}^{u} \lambda_*^{-k_i-l_i} P_i(m,n) + \frac{\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} P_i(m+1,n)}{1 - \sum_{i=1}^{u} \lambda_*^{1-k_i-l_i}(P_i(m+2,n) + P_i(m+1,n+1))}$$

$$+ \frac{\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} P_i(m,n+1)}{1 - \sum_{i=1}^{u} \lambda_*^{1-k_i-l_i}(P_i(m+1,n+1) + P_i(m,n+2))} \Big) > 1. \tag{21}$$

Then the conclusions of Theorem 1 *hold.*

Proof. In fact, from (18) and (19), we have

$$A_{m+1,n} \ge \sum_{i=1}^{u} P_i(m+1,n)(\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m,n} \quad \text{for } m \ge m_l,\ n \ge n_l. \tag{22}$$

and

$$A_{m,n+1} \ge \sum_{i=1}^{u} P_i(m,n+1)(\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m,n} \quad \text{for } m \ge m_l, n \ge n_l. \tag{23}$$

Hence,

$$A_{m,n} = A_{m+1,n} + A_{m,n+1} + \sum_{i=1}^{u} P_i(m,n) A_{m-k_i,n-l_i}$$

$$\ge \sum_{i=1}^{u} (P_i(m+1,n) + P_i(m,n+1))(\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m,n} + \sum_{i=1}^{u} P_i(m,n)(\lambda_* + \varepsilon_l)^{1-k_i-l_i} A_{m-1,n},$$

and

$$A_{m,n} \ge \frac{\sum_{i=1}^{u} (\lambda_* + \varepsilon_l)^{1-k_i-l_i} P_i(m,n)}{1 - \sum_{i=1}^{u} (\lambda_* + \varepsilon_l)^{1-k_i-l_i}(P_i(m+1,n) + P_i(m,n+1))} A_{m-1,n}. \tag{24}$$

Similarly,

$$A_{m,n} \ge \frac{\sum_{i=1}^{u} (\lambda_* + \varepsilon_l)^{1-k_i-l_i} P_i(m,n)}{1 - \sum_{i=1}^{u} (\lambda_* + \varepsilon_l)^{1-k_i-l_i}(P_i(m+1,n) + P_i(m,n+1))} A_{m,n-1}. \tag{25}$$

Substituting the above inequalities into (20) and letting $l \to \infty$, we obtain a contradiction with (21). The proof is complete. □

Since

$$\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} P_i(m+1,n) \geq \lambda_* \sum_{i=1}^{u} \lambda_*^{-k_i-l_i} p_i = \lambda_*(1-\lambda_*),$$

from (21), we can obtain a simpler condition.

Corollary 1. *Assume that that* (2) *holds. Further assume that*

$$\limsup_{m,n\to\infty} \sum_{i=1}^{u} \lambda_*^{-k_i-l_i} P_i(m,n) > 2 - \frac{1}{\lambda_*^2 + (1-\lambda_*)^2}. \tag{26}$$

Then the conclusions of Theorem 1 *hold.*

In fact, (26) implies (21).

Theorem 3. *Assume that* (2) *holds. Further assume that*

$$\limsup_{m,n\to\infty} \Big(\frac{1}{(1-\lambda_*)^2} \sum_{i=1}^{u} P_i(m,n) \lambda_*^{-k_i-l_i} (1-\lambda_*^{l_i+1})(1-\lambda_*^{k_i+1}) + Q(m,n,\lambda_*) \Big) > 1, \tag{27}$$

where λ_* *is the largest root of* (6) *on* $(0,1]$ *and*

$$Q(m,n,\lambda_*) = \frac{\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} P_i(m+1,n+1)}{1 - \sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} (P_i(m+2,n+1) + P_i(m+1,n+2))} \times \frac{\sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} P_i(m+1,n)}{1 - \sum_{i=1}^{u} \lambda_*^{1-k_i-l_i} (P_i(m+2,n) + P_i(m+1,n+1))}.$$

Then:

(i) (7) *has no eventually positive solutions;*
(ii) (8) *has no eventually negative solutions; and*
(iii) *every solution of Eq.* (1) *oscillates.*

Proof. It is easy to prove (ii) and (iii) similarly to the proof of (i). Assume, for the sake a contradiction, that $\{A_{m,n}\}$ is an eventually positive solution of (1). Summing (1) in n from n ($\geq n_1$) to ∞, we have

$$\sum_{v=n}^{\infty} A_{m+1,v} - A_{m,n} + \sum_{i=1}^{u} \sum_{v=n}^{\infty} P_i(m,v) A_{m-k_i,v-l_i} \leq 0.$$

We rewrite the above inequality in the form

$$\sum_{v=n+1}^{\infty} A_{m+1,v} + A_{m+1,n} - A_{m,n} + \sum_{i=1}^{u} \sum_{v=n}^{\infty} P_i(m,v) A_{m-k_i,v-l_i} \leq 0.$$

Summing this inequality in m from m ($\geq m_1$) to ∞, we obtain

$$\sum_{s=m}^{\infty} \sum_{v=n+1}^{\infty} A_{s+1,v} - A_{m,n} + \sum_{i=1}^{u} \sum_{s=m}^{\infty} \sum_{v=n}^{\infty} P_i(s,v) A_{s-k_i,v-l_i} \leq 0. \tag{28}$$

From (24) and (25), we have

$$A_{m+1,n+1} \geq Q(m,n,\lambda_* + \varepsilon_l) A_{m,n}. \tag{29}$$

By (28), we get

$$\begin{aligned} A_{m,n} &\geq \sum_{s=m}^{\infty}\sum_{v=n+1}^{\infty} A_{s+1,v} + \sum_{i=1}^{u}\sum_{s=m}^{\infty}\sum_{v=n}^{\infty} P_i(s,v) A_{s-k_i,v-l_i} \\ &\geq A_{m+1,n+1} + \sum_{i=1}^{u}\sum_{s=m}^{m+k_i}\sum_{v=n}^{n+l_i} P_i(s,v) A_{s-k_i,v-l_i} \\ &\geq A_{m+1,n+1} + \sum_{i=1}^{u}\sum_{s=0}^{k_i}\sum_{v=0}^{l_i} P_i(s+m,v+n) A_{m+s-k_i,n+v-l_i} \\ &\geq A_{m+1,n+1} + A_{m,n}\sum_{i=1}^{u}\sum_{s=0}^{k_i}\sum_{v=0}^{l_i} P_i(s+m,v+n)(\lambda_* + \varepsilon_l)^{(s-k_i)+(v-l_i)} \\ &= A_{m+1,n+1} + A_{m,n}\sum_{i=1}^{u}\Big(\sum_{s=m}^{m+k_i}\sum_{v=n}^{n+l_i} P_i(m,n)(\lambda_* + \varepsilon_l)^{(s-m-k_i)+(v-n-l_i)}\Big). \end{aligned}$$

Letting $l \to \infty$, the above two inequalities imply

$$\limsup_{m,n\to\infty}\Big(\frac{1}{(1-\lambda_*)^2}\sum_{i=1}^{u} P_i(m,n)\lambda_*^{-k_i-l_i}(1-\lambda_*^{l_i+1})(1-\lambda_*^{k_i+1}) + Q(m,n,\lambda_*)\Big) \leq 1,$$

which contradicts (27) and completes the proof. □

Since

$$Q(m,n,\lambda_*) \geq (\frac{\lambda_*(1-\lambda_*)}{1-2\lambda_*+2\lambda_*^2})^2,$$

we can derive a simpler condition from (27).

Corollary 2. (27) *is replaced by*

(30)

$$\limsup_{m,n\to\infty}\frac{1}{(1-\lambda_*)^2}\sum_{i=1}^{u} P_i(m,n)\lambda_*^{-k_i-l_i}(1-\lambda_*^{l_i+1})(1-\lambda_*^{k_i+1}) > 1 - \Big(\frac{\lambda_*(1-\lambda_*)}{1-2\lambda_*+2\lambda_*^2}\Big)^2,$$

then the conclusions of Theorem 3 *hold.*

In the following, we consider the nonlinear partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + \sum_{i=1}^{u} P_i(m,n) f_i(A_{m-k_i,n-l_i}) = 0, \quad m \geq 0, n \geq 0, \tag{31}$$

where

$$f \in (R,R) \quad \text{and } u f_i(u) > 0 \quad \text{for } u \neq 0. \tag{32}$$

Theorem 4. *Assume that* (2) *holds and*

$$\liminf_{u\to 0}\frac{f_i(u)}{u}\geq 1,\qquad \text{for } i=1,2,\dots,u. \tag{33}$$

Further assume that (4) *has positive roots and one of conditions* (10), (21) *or* (27) *holds. Then every solution of Eq.* (31) *oscillates.*

Proof. Assume, for the sake a contradiction, that Eq. (31) has a non-oscillatory solution $\{A_{m,n}\}$. We assume that $\{A_{m,n}\}$ is eventually positive. The case where $\{A_{m,n}\}$ is eventually negative is similar and is omitted. It is not difficult to see that [1]

$$\lim_{m,n\to\infty} A_{m,n}=0. \tag{34}$$

By (33) and (34), we get

$$\liminf_{n\to\infty}\frac{f_i(A_{m-k_i,n-l_i})}{A_{m-k_i,n-l_i}}\geq 1,\qquad \text{for } i=1,2,\dots,u. \tag{35}$$

From (31) and (33), we have

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+\sum_{i=1}^{u}P_i(m,n)A_{m-k_i,n-l_i}\leq 0,\quad m\geq 0,\ n\geq 0. \tag{36}$$

But by Theorems 1-3, when (2) and one of conditions (10), (21), or (27) hold, (36) cannot have eventually positive solutions. This contradiction completes the proof. □

Example 1. Consider the partial difference equation

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+P(m,n)A_{m-1,n-1}=0,\quad m\geq 0,\ n\geq 0, \tag{37}$$

where

$$P(m,n)=\begin{cases}\frac{1}{27}, \text{if} & m=n\in N_0,\\ \frac{1}{5}, & \text{otherwise.}\end{cases}$$

For equation (37), (9) becomes

$$2\lambda-1+\frac{1}{27}\lambda^{-2}=0, \tag{38}$$

which has a positive root $\lambda=\frac{1}{3}$. The limiting equation of (37) is

$$A_{m+1,n}+A_{m,n+1}-A_{m,n}+\frac{1}{27}A_{m-1,n-1}=0,\quad m\geq 0,\ n\geq 0, \tag{39}$$

which has a positive solution $\{A_{m,n}\}=\{\frac{1}{3^{m+n}}\}$ $m,n\in N_0$. Equation (6) becomes the form

$$\lambda-1+\frac{1}{27}\lambda^{-2}=0,$$

which has a positive root $\lambda_*=\frac{2}{3}\cos\frac{\phi}{3}$, where $\cos\phi=\frac{1}{2}$. Thus, $\phi=\frac{\pi}{3}$ and $\lambda_*\approx 0.63$. Since $\limsup_{m,n\to\infty}P(m,n)=\frac{1}{5}$, we have

$$\limsup_{m,n\to\infty}P(m,n)>\lambda_*^2\Big(2-\frac{1}{\lambda_*^2+(1-\lambda_*)^2}\Big)\approx 0.05.$$

By Corollary 1, every solution of (37) is oscillatory, but none of the results in the literature (see survey paper [8]) can be applied to this equation.

REFERENCES

[1] W. G. Kelley and A. C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, New York, (1991).

[2] Sui Sun Cheng, Partial Difference Equations, Taylor & Francis, London, (2003).

[3] R. Courant, K. Fridrichs and H. Lewye, On partial difference equations of mathematical physics, IBM J. 11, 215–234, (1967).

[4] Paul Meakin, Models for material failure and deformation, Science, 252 (4), (1991), 226–234.

[5] B. E. Shi and L. O. Chua, Resistive grid image filtering:input / output analysis via the CNN framework, IEEE Trans. Circuits Syst.I, 39 (7), (1999) , 531–548.

[6] B. G. Zhang and S. T. Liu, Necessary and sufficient conditions for oscillations of partial difference equations, DCDIS, 3 (1997), 89–96.

[7] B. G. Zhang and J. S. YU, Linearized oscillation theorems for certain nonlinear delay partial difference equations, Comput. Math. Appl. 35 (1998), 111–116.

[8] B. G. Zhang and R. P. Agarwal, The oscillation and stability of delay partial difference equations, Comput. Math. Appl. 45 (2003), 1253–1295.

[9] B. G. Zhang and S. T. Liu, On the oscillation of two partial difference equations, J. Math. Anal. Appl. 206 (1997), 480–492.

[10] C. J. Tian and B. G. Zhang, Frequent oscillation of a class of partial difference equations, Zeit. Anal. Anwend. 18 (1), 111–130, 1999.

[11] B. G. Zhang and Jian She Yu, Comparison and linearized oscillation theorems for a nonlinear partial difference equations, ANZIAM J. 42 (2001), 1–9.