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OPTIMIZATION OF THE MOTION OF A VISCO-ELASTIC FLUID VIA MULTIVALUED TOPOLOGICAL DEGREE METHOD

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ABSTRACT. We consider the application of the topological degree theory for noncompact multivalued vector fields to the problem of existence of an optimal feedback control in the presence of delay for the model of the motion of a visco-elastic fluid satisfying the Voight rheological relation. The notion of a weak solution to the problem is introduced and the operator treatment of the problem allows to reduce it to the existence of a fixed point for a certain condensing multivalued map. We give an a priori estimate for solutions of the problem and the use of the degree method allows to prove the non-voidness and compactness of the solution set. As the result we obtain the existence of a solution minimizing the given quality functional.

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INTRODUCTION

Problems of optimal control in hydrodynamics have attracted the attention of many researchers (see, for example, monographs [2], [6], [7] and the bibliography therein). In the most of these works the authors considered the fluids described by Navier–Stokes type equations. In some recent papers methods of multivalued analysis were used to obtain certain optimization results in problems of control of the motion for non-Newtonean visco-elastic fluids (see [8], [13]).

In the present paper we apply the topological degree theory for condensing multivalued vector fields to prove the existence of an optimal feedback control in the Voight model of a visco-elastic fluid. This model describes media which need some time to start the motion under the action of an applied external force (see, e.g., [12]). It should be mentioned that the solvability of the initial – boundary value problem in the Voight model (and generalized Voight-Kelvin model) was initially studied in the works of A. P. Oskolkov [9], [10]. In our paper we consider the situation when the external force is the result of the joint action of the force which is the subject to

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a delay feedback control depending on the velocity of the fluid and the other force which is determined by the velocity of the fluid and its derivative with respect to time.

The paper is organized in the following way. In the first section we present necessary notions and definitions from the theory of multivalued maps, measures of noncompactness, and condensing maps. In the second section we are describing the model, introduce the notion of a weak solution to the problem, and give the formulation of our main result. The next section is devoted to the operator treatment of the problem, where finally the problem under consideration is reduced to the existence of a fixed point for a condensing multivalued map. In the last section we give an a priori estimate for solutions of the problem, allowing to apply the multivalued topological degree methods to verify the non-voidness and compactness of the solution set. As the result we obtain the existence of a solution minimizing the given quality functional, proving the main result of the paper.

1. **PRELIMINARIES**

We will suppose that the considered fluid fills a container with rigid walls modelled by a bounded domain Ω in $\mathbb{R}^n$, $n \in \{2,3\}$, the boundary $\partial\Omega$ is supposed to be Lipschitz. The density of the fluid is supposed to be constant and equal to one.

By X^* we will denote the space dual to a real Banach space X and by $\langle \mathbf{g}, \mathrm{y}\rangle$ we denote the action of the functional $\mathbf{g} \in X^*$ on the element $\mathrm{y} \in X$, the scalar product in an arbitrary Hilbert space $\mathcal{H}$ is denoted by $(\cdot,\cdot)_{\mathcal{H}}$.

We will use $L_p(\Omega)^n$, $W_2^1(\Omega)^n$ as the standard notation of Lebesgue and Sobolev spaces of functions defined on Ω with the values in $\mathbb{R}^n$. Let $\mathfrak{D}(\Omega)^n$ be the space of of functions of the class C^∞ with the values in $\mathbb{R}^n$ and a compact support in Ω and

$$\mathfrak{D}_s(\Omega)^n = \{v \in \mathfrak{D}(\Omega)^n : \operatorname{div} v = 0\}.$$

We will consider also the following spaces (see, for example, [11]):

The closures of $\mathfrak{D}_s(\Omega)^n$ in the norms of the spaces $L_2(\Omega)^n$ and $W_2^1(\Omega)^n$ will be denoted respectively by H and V. The space V is Hilbert with the scalar product defined by $(\varphi,\psi)_V = \sum_{i=1}^n \left(\frac{\partial\varphi}{\partial x_i}, \frac{\partial\psi}{\partial x_i}\right)_{L_2(\Omega)^n}$.

We use the notation $C([a,b];X)$ and $C^1([a,b];X)$ for the spaces of continuous and continuously differentiable functions, respectively, defined on the interval $[a,b]$ with the values in X. We consider the usual norm in the space $C([a,b];X)$:

$$\|v\|_{C([a,b],X)} = \max_{t\in[a,b]} \|v(t)\|_X, \tag{1.1}$$

For $T>0$, let us introduce the following notation:

$$C = C([0,T],V), \quad C_* = C([0,T],V^*).$$

For a given $h > 0$, by the symbol C^1 we will denote a Banach space of functions $v \in C([-h,T];V)$ such that $\tilde{v} = v|_{[0,T]} \in C^1([0,T];V)$ with the norm

$$\|v\|_{C^1} = \|v\|_{C([-h,T];V)} + \|\tilde{v}\,'\|_{C([0,T];V)}.$$

As one of the main tools in our studies we will use some aspects of the theory of multivalued maps.

Let us briefly describe necessary preliminaries (see, e.g. [1], [4] for details).

Let X and Y be metric spaces and $K(Y)$ denote the collection of all nonempty compact subsets of Y.

A multivalued map (multimap) $F : X \to K(Y)$ is said to be:

(i) *upper semicontinuous* (u.s.c.) if the small pre-image $F_+^{-1}(W) = \{x \in X : F(x) \subset W\}$ of each open subset $W \subset Y$ is open in X;

(ii) *closed* if its graph is a closed subset of the space $X \times Y$;

(iii) *compact* if its range $F(X)$ is relatively compact in Y;

(iv) *completely continuous* if it is u.s.c. and the image $F(D)$ of each bounded set $D \subset X$ is relatively compact in Y.

We recall that an u.s.c. multimap $F : X \to K(Y)$ is closed.

Proposition 1.1. A closed and locally compact multimap $F : X \to K(Y)$ is u.s.c.

Proposition 1.2. If a multimap $F : X \to K(Y)$ is u.s.c. then the image $F(M)$ of each compact set $M \subset X$ is compact.

A function $\chi : 2^X \to [0,+\infty]$,

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \quad \text{has a finite} \quad \varepsilon - \text{net}\}$$

is said to be *the Hausdorff measure of noncompactness (MNC)* in X.

It is clear that the Hausdorff MNC is *regular*, i.e., the equality $\chi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

Let us mention also the following properties (see [4]).

Proposition 1.3. If X and Y are normed spaces with the Hausdorff MNCs χ_0 and χ_1 respectively and $L : X \to Y$ is a bounded linear operator then

$$\chi_1(L\Omega) \leq \|L\|\,\chi_0(\Omega)$$

for each bounded set $\Omega \subset X$.

Proposition 1.4. Let E be a separable Banach space; a multifunction $G : [a,b] \to 2^E$ be integrably bounded and admit an integrable selection. If

$$\chi(G(t)) \leq q(t)$$

for a.e. $t \in [a, b]$, where $q(\cdot) \in L_1[a, b]$ then

$$\chi\left(\int_a^t G(s)ds\right) \leq \int_a^t q(s)ds,$$

where $\int_a^t G(s)ds := \{\int_a^t g(s)ds : g \in L_1[0, t]; g(\tau) \in G(\tau) \text{ for a.e. } \tau \in [0, t]\}$

Let χ_0 and χ_1 be the Hausdorff MNCs in metric spaces X and Y respectively. For some $k \geq 0$, a multimap $F : X \to K(Y)$ is said to be (k, χ_0, χ_1)–*bounded* provided

$$\chi_1(F(\Omega)) \leq k\, \chi_0(\Omega)$$

for each bounded $\Omega \subset X$.

In case $X = Y$ and $0 \leq k < 1$ we will say that F is (k, χ)–*condensing* or simply *condensing* (with respect to χ).

As a particular case of Proposition 2.2.2 of [4] we have the following statement.

Proposition 1.5. Let E_0, E_1 be Banach spaces; χ_0, χ_1 the Hausdorff MNCs in E_0 and E_1 respectively; $X \subset E_0$ a bounded closed subset. If a multimap $F : X \to K(E_1)$ is compact and a map $f : X \to E$ is k-Lipschitz, $k \geq 0$ then their sum $F + f : X \to K(E_1)$ is (k, χ_0, χ_1)–bounded.

Now, let E be a Banach space, $Kv(E)$ denote the collection of all nonempty compact convex subsets of E and $U \subset E$ be an open bounded set.

Suppose that an u.s.c. multimap $F : \overline{U} \to Kv(E)$ is condensing and the fixed points set $Fix\, F = \{x \in \overline{U} : x \in F(x)\}$ does not intersect the boundary ∂U. In this situation the integer characteristic, the topological degree $deg(i - F, \overline{U})$ of the corresponding multifield $\Phi = i - F$, $\Phi(x) = x - F(x)$, is well defined and possesses the standard properties among which we will select the following.

a) *The homotopy invariance.* Let $G : \overline{U} \times [0, 1] \to Kv(E)$ be an u.s.c. (k, χ)–condensing family in the sense that

$$\chi(G(\Omega) \times [0, 1]) \leq k\, \chi(\Omega)$$

for each $\Omega \subset \overline{U}$. Suppose that $x \notin G(x, \lambda)$ for all $(x, \lambda) \in \partial U \times [0, 1]$. Then

$$deg(i - G(\cdot, 0), \overline{U}) = deg(i - G(\cdot, 1), \overline{U}) \ .$$

b) *The fixed point property.* If $deg(i - F, \overline{U}) \neq 0$ then $\emptyset \neq Fix\, F \subset U$.

2. THE STATEMENT OF THE PROBLEM AND THE FORMULATION OF THE MAIN RESULT

It is well known that the motion of a fluid is described by the following system of differential equations in the Cauchy form

$$\frac{\partial v}{\partial t}(t,x) + \sum_{i=1}^{n} v_i(t,x)\frac{\partial v}{\partial x_i}(t,x) + \operatorname{grad} p\,(t,x) - \operatorname{Div}\sigma(t,x) = \\ = f(t,x), \quad (t,x) \in Q_T\,. \tag{2.1}$$

In this relation we use the following notation: t is the time parameter varying on the interval $[0,T]\,, T>0$; $Q_T = [0,T]\times\Omega$; $x = (x_1,\dots,x_n)\in\Omega$; $v = (v_1,\dots,v_n)$ is the field of velocities of fluid particles; p is the function of the pressure of the fluid, $\operatorname{grad} p = \left(\frac{\partial p}{\partial x_1},\dots,\frac{\partial p}{\partial x_n}\right)$; f is the vector of the density of external forces exerted on the fluid; σ is the deviator of the tensions tensor, $\operatorname{Div}\sigma = \left(\sum\limits_{j=1}^{n}\frac{\partial\sigma_{1j}}{\partial x_j}, \sum\limits_{j=1}^{n}\frac{\partial\sigma_{2j}}{\partial x_j},\dots,\sum\limits_{j=1}^{n}\frac{\partial\sigma_{nj}}{\partial x_j}\right)$.

It will be supposed that the fluid under consideration is subjected to the Voight rheological relation

$$\sigma = 2\mu_1\mathcal{E} + 2\mu_2\frac{\partial\mathcal{E}}{\partial t}, \quad \text{where } \mu_1,\mu_2 = const, \text{ with } \mu_2>0, \tag{2.2}$$

and $\mathcal{E} = \mathcal{E}(v) = \big(\mathcal{E}_{ij}(v)\big)$ is the tensor of velocities of deformations:

$$\mathcal{E}_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

After the substitution of relation (2.2) into equation (2.1) it takes the form

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i\frac{\partial v}{\partial x_i} + \operatorname{grad} p - \mu_1\triangle v - \mu_2\frac{\partial\triangle v}{\partial t} = f. \tag{2.3}$$

We will assume that the action of external forces f on the fluid can be represented as the sum of two vectors,

$$f = f_1 + f_2,$$

where f_1 is the controlled function which is chosen at each moment $t\in[0,T]$ in accordance with a multivalued feedback relation with the time delay and f_2 describes a certain natural force depending on the distribution of velocities v and their derivatives with respect to time.

To be more exact, denote $\mathcal{C} = C([-h,0];V)$ and let E be a Banach space of control parameters. We will assume that a feedback multimap $U:[0,T]\times\mathcal{C}\to Kv(E)$ satisfies the following conditions:

$(U1)$ the multifunction $U(\cdot,c):[0,T]\to Kv(E)$ has a strongly measurable selection for each $c\in\mathcal{C}$;

$(U2)$ the multimap $U(t,\cdot):\mathcal{C}\to Kv(E)$ is u.s.c. for a.e. $t\in[0,T]$;

$(U3)$ the multimap U is globally bounded, i.e., there exists a constant $M > 0$ such that

$$\|U(t,c)\| := \sup\{\|u\| \ : \ u \in U(t,c)\} \leq M$$

for a.e. $t \in [0,T]$ and all $c \in \mathcal{C}$;

$(U4)$ for each bounded $D \subset \mathcal{C}$, the set $U(t,D)$ is relatively compact for a.e. $t \in [0,T]$.

Let us note that conditions $(U1)$–$(U3)$ imply that for every $v \in C^1$ the set

$$\mathcal{P}_U(v) = \{u \in L_1([0,T];E) : \ u(t) \in U(t,v_t) \text{ for a.e. } t \in [0,T]\}$$

is non-empty (see, e.g. [1], [4]). Here $v_t \in \mathcal{C}$ denotes, for $t \in [0,T]$, the function $v_t(\theta) = v(t+\theta)$, $\theta \in [-h,0]$.

Now we assume that, under a given control $u \in \mathcal{P}_U(v)$, the force $f_1(u) \in C_*$ is defined by the Bochner integral relation

$$f_1(u)(t) = \int_0^t K(t,s)u(s)ds.$$

Here $K(t,s) : E \to V^*$ is a bounded linear operator for each $(t,s) \in \Delta = \{(t,s) \in [0,T] \times [0,T] : t \geq s\}$. It is supposed also that the kernel K satisfies the following conditions:

$(K1)$ the function K is strongly continuous in the sense that $(t,s) \in \Delta \to K(t,s)u \in V^*$ is a continuous function for each $u \in E$;

$(K2)$ the function K is uniformly bounded, i.e., there exists $N > 0$ such that

$$\|K(t,s)\| \leq N$$

for all $(t,s) \in \Delta$.

Since $f_1 : L_1([0,T];E) \to C_*$ is the linear continuous operator, as the direct consequence of Theorem 1.5.30 of [1], we obtain the following statement.

Lemma 2.1. *The multimap $f_1 \circ \mathcal{P}_U$ is closed.*

Moreover, taking into account property $(U4)$ and Proposition 1.4 and applying the Arzelá–Ascoli theorem we can conclude that the image $f_1 \circ \mathcal{P}_U(\Omega)$ of each bounded set $\Omega \subset C^1$ is relatively compact. Finally, using Proposition 1.1 we have the following assertion.

Lemma 2.2. *The multimap $f_1 \circ \mathcal{P}_U$ has compact convex values and is completely continuous.*

Concerning the vector of forces f_2 it will be assumed that, at each moment $t \in [0,T]$ and at every point $x \in \Omega$ it depends on the velocity of the fluid $v(t,x)$ and its derivative $\frac{\partial v}{\partial t}(t,x)$:

$$f_2(t,x) = \xi(v(t,x), \frac{\partial v}{\partial t}(t,x)), \tag{2.4}$$

where the function $\xi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following Lipschitz type condition:

$$\|\xi(\alpha, \beta) - \xi(\overline{\alpha}, \overline{\beta})\| \leq l\,(\|\alpha - \overline{\alpha}\| + \|\beta - \overline{\beta}\|)$$

for some $l > 0$.

It is easy to see that relation (2.4) generates a continuous map (which will be denoted, for simplicity, by the same symbol) $f_2 : C^1 \to C_*$ satisfying the Lipschitz condition

$$(f_2 1) \qquad \|f_2(v) - f_2(\tilde{v})\|_{C_*} \leq l\ \|v - \tilde{v}\|_{C^1}$$

with the same constant l.

It will be supposed also that

$(f_2 2)$ the function f_2 is bounded, i.e., there exists $L > 0$ such that

$$\|f_2(v)\|_{C_*} \leq L$$

for all $v \in C^1$.

So, the total action of external forces on the fluid is described by the multimap $\mathfrak{F} : C^1 \to Kv(C_*)$,

$$\mathfrak{F}(v) = f_1 \circ \mathcal{P}_U(v) + f_2(v).$$

Remark 2.1. Let us mention that from conditions $(U3)$, $(K2)$, and $(f_2 2)$ it follows that the multimap $\mathfrak{F}$ is bounded, i.e., there exists a constant $\mathcal{N} > 0$ such that

$$\|f\|_{C_*} \leq \mathcal{N}$$

for each $f \in \mathfrak{F}(v)$ and $v \in C^1$.

Remark 2.2. From Lemma 2.2 and the properties of multivalued maps (see, e.g. [1], [4]) it follows that the multimap $\mathfrak{F}$ is u.s.c.

Take an initial function $\nu \in \mathcal{C}$ satisfying the boundary condition of adhesion

$$\nu|_{[-h,0]\times\partial\Omega} = 0.$$

If we deal with the problem of control over the motion of the considered fluid in the class of strong solutions, it may be formulated in the following way: to find a collection of functions (v, σ, p, f) satisfying relations (2.2), (2.3), the feedback condition

$$(2.5) \qquad f \in \mathfrak{F}(v),$$

the incompressibility condition

$$(2.6) \qquad \operatorname{div} v = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0,$$

the initial condition

$$(2.7) \qquad v_0 = \nu,$$

and the boundary condition

$$v|_{[0,T]\times\partial\Omega} = 0. \tag{2.8}$$

We will consider problem (2.2), (2.3), (2.5) – (2.8) for the class of weak solutions.

Definition 2.3. A pair of functions $(v, f) \in C^1 \times C_*$ is called a weak solution to problem (2.2), (2.3), (2.5) – (2.8) if it satisfies feedback condition (2.5) and the equality

$$(v'(t), \varphi)_{L_2(\Omega)^n} + \mu_2(v'(t), \varphi)_V + \mu_1(v(t), \varphi)_V - \\ - \sum_{i,j=1}^{n} \left(v_i(t)v_j(t), \frac{\partial \varphi_j}{\partial x_i} \right)_{L_2(\Omega)} = \langle f(t), \varphi \rangle \tag{2.9}$$

for an arbitrary function $\varphi \in V$ and each $t \in [0, T]$ whereas the function v obeys also initial condition (2.7).

Let us note that the above interpretation of a weak solution is in agreement with the corresponding notion being widely used in hydrodynamics (see, e.g. [11]). If a pair (v, f) represents a strong solution to (2.2), (2.3), (2.5) – (2.8) we can come to relation (2.9) by means of a scalar multiplication of equation (2.3) in $L_2(\Omega)^n$ by a given element $\varphi \in V$ and through the integrating in parts. It should be mentioned that under this operation we have $(grad\, p, \varphi)_{L_2(\Omega)^n} = 0$.

Furthermore, our goal is to solve the following optimization problem.

Let $\Sigma \subset C^1 \times C_*$ be the set of all weak solutions to problem (2.1) — (2.8). Suppose that $j : \Sigma \to \mathbb{R}$ is a lower semicontinuous, bounded from below quality functional.

For example, j may have the form

$$j(v, f) = C_1 \int_\Omega |v(x, T) - \bar{v}(x)|^2 \, dx + C_2 \int_0^T \|(f - f_2(v))(t)\|_{V^*} \, dt$$

where the first term characterizes the deviation of velocities at the final moment from a given distribution (e.g. $\bar{v} \equiv 0$) and the second term estimates the cost of control.

The main result of this paper is the following statement.

Theorem 2.4. *Under above conditions, if the Lipschitz constant l from condition$(f_2 1)$ is sufficiently small then there exists a weak solution (v^*, f^*) to problem (2.1) — (2.8) such that*

$$j(v^*, f^*) = \inf_{(v,f)\in\Sigma} j(v, f). \tag{2.10}$$

The proof of this assertion will be given in Section 4.2.

3. THE OPERATOR TREATMENT OF THE PROBLEM

Let us note (see [11]) that we have the following continuous embeddings:

$$V \subset H \equiv H^* \subset V^*, \tag{3.1}$$

where $H \equiv H^*$ is the identification of the spaces H and H^* which follows from the Riesz theorem on the representation of a linear functional in a Hilbert space.

Embeddings (3.1) generate the inclusion map $\widehat{J} : V \to V^*$ possessing the following properties:

Lemma 3.1. 1. *The operator* $\widehat{J} : V \to V^*$ *is linear and continuous.*

2. *The operator* J *defined on* C *by the equality*

$$J[v](t) = \widehat{J}[v(t)] \quad \textit{for } v \in C,\ t \in [0, T],$$

acts into C_*, *it is linear and continuous.*

The following statement is actually the corollary of the above-mentioned Riesz theorem.

Lemma 3.2. 1. *For every* $\psi \in V$, *the linear continuous form*

$$\varphi \in V \mapsto (\psi, \varphi)_V,$$

defines the operator $\widehat{A} : V \to V^*$ *such that*

$$\langle \widehat{A}\psi, \varphi\rangle = (\psi, \varphi)_V.$$

Moreover, the operator $\widehat{A}$ *is linear continuous, has a continuous inverse, and satisfies the relation*

$$\|\widehat{A}\psi\|_{V^*} = \|\psi\|_V \quad \forall \psi \in V. \tag{3.2}$$

2. *The operator* A *defined on* C *by the equality*

$$A[v](t) = \widehat{A}[v(t)] \quad \textit{for } v \in C,\ t \in [0, T],$$

acts into C_*, *it is linear continuous, has a continuous inverse, and satisfies the relation*

$$\|Av\|_{C_*} = \|v\|_C \quad \forall v \in C. \tag{3.3}$$

The operators $\mu_2\widehat{A} + \widehat{J}$ and $\mu_2 A + J$ possess the following properties:

Lemma 3.3. 1. *The operator* $\mu_2\widehat{A} + \widehat{J} : V \to V^*$ *is linear continuous, has a continuous inverse, and obeys the estimates*

$$\mu_2\|\psi\|_V \le \|(\mu_2\widehat{A} + \widehat{J})\psi\|_{V^*} \le c_1\|\psi\|_V \quad \forall \psi \in V, \tag{3.4}$$

where the constant c_1 *does not depend on* ψ.

2. *The operator* $\mu_2 A + J : C \to C_*$ *is linear continuous, has a continuous inverse, and satisfies the estimates*

$$\mu_2\|v\|_C \le \|(\mu_2 A + J)v\|_{C_*} \le c_1\|v\|_C \quad \forall v \in C. \tag{3.5}$$

Proof. 1) The relations

$$\langle(\mu_2\widehat{A} + \widehat{J})\psi, \psi\rangle = \mu_2(\psi,\psi)_V + (\psi,\psi)_H = \mu_2\|\psi\|_V^2 + \|\psi\|_H^2 \ge \mu_2\|\psi\|_V^2 \tag{3.6}$$

imply the required lower estimate.

To obtain the upper estimate, at first let us mention that the embedding $V \subset H$ yields the inequality

$$\|\varphi\|_H \le \kappa\|\varphi\|_V \tag{3.7}$$

for a certain constant κ which does not depend on φ. Further, applying the Cauchy–Schwarz inequality we have:

$$\|(\mu_2\widehat{A} + \widehat{J})\psi\|_{V^*} \stackrel{\text{def}}{=} \sup_{\substack{\varphi\in V\\ \|\varphi\|_V=1}} |\langle(\mu_2\widehat{A} + \widehat{J})\psi, \varphi\rangle| =$$
$$= \sup_{\substack{\varphi\in V\\ \|\varphi\|_V=1}} |\,\mu_2(\psi,\varphi)_V + (\psi,\varphi)_H| \le \sup_{\substack{\varphi\in V\\ \|\varphi\|_V=1}} (\,\mu_2\|\psi\|_V + \kappa\|\psi\|_H)\|\varphi\|_V = c_1\|\psi\|_V,$$

where $c_1 = \mu_2 + \kappa^2$.

Now let us demonstrate that the operator $\mu_2\widehat{A} + \widehat{J}$ has a continuous inverse. In fact, the above inequalities imply that we may define a new scalar product on the space V by the equality

$$(\psi,\varphi)_{\widetilde{\mathbf{V}}} = \langle[\mu_2\widehat{A} + \widehat{J}]\psi, \varphi\rangle$$

and then the assertion follows from the same Riesz theorem.

2) The results of this item follow from the previous one and the Banach theorem on the inverse operator. □

For each fixed $\psi \in V$ the form

$$\varphi \in V \mapsto \sum_{i,j=1}^{n} \int_\Omega \psi_i\psi_j \frac{\partial\varphi_j}{\partial x_i}\,dx$$

is linear and continuous on V. Therefore we may define the operator $\widehat{B} : V \to V^*$,

$$\langle\widehat{B}(\psi), \varphi\rangle = \sum_{i,j=1}^{n} \int_\Omega \psi_i\psi_j\,\frac{\partial\varphi_j}{\partial x_i}\,dx \quad \text{for } \psi,\ \varphi \in V.$$

We will need the following statement.

Lemma 3.4. 1. *The operator* $\widehat{B}: V \to V^*$ *satisfies the relation*

$$(3.8)\quad \|\widehat{B}(\varphi_1) - \widehat{B}(\varphi_2)\|_{V^*} \leq \\ \leq c_2 \left(\|\varphi_1\|_{L_4(\Omega)^n} + \|\varphi_2\|_{L_4(\Omega)^n}\right) \|\varphi_1 - \varphi_2\|_{L_4(\Omega)^n} \quad \forall \varphi_1, \varphi_2 \in V,$$

where the constant c_2 *does not depend on* φ_1 *and* φ_2*.*

2. *The operator* B *defined on* C^1 *by the equality*

$$(3.9)\qquad B[v](t) = \widehat{B}[v(t)] \qquad \text{for } v \in C^1,\ t \in [0,T],$$

acts into C_* *and is completely continuous.*

Proof. 1) See [5].

2) Inequality (3.8) implies

$$(3.10)\quad \|B(v) - B(w)\|_{C_*} \leq \\ \leq c_2 \left(\|\widetilde{v}\|_{C([0,T],L_4(\Omega)^n)} + \|\widetilde{w}\|_{C([0,T],L_4(\Omega)^n)}\right) \|\widetilde{v} - \widetilde{w}\|_{C([0,T],L_4(\Omega)^n)}$$

for all $v, w \in C^1$, where $\widetilde{v}, \widetilde{w}$ denote the restrictions of v and w to $[0,T]$.

Since $B(0) = 0$, the above estimate verifies that the operator B actually has its range in C_*, moreover, it obviously implies the continuity of B.

Let us demonstrate that the operator B is completely continuous. Suppose $\{v_m\}_{m=1}^\infty$ is a bounded sequence in C^1. Clearly $B(v_m) = B(\widetilde{v}_m)$, $m = 1, 2, \ldots$, where $\widetilde{v}_m = v_m|_{[0,T]} \in C^1([0,T]; V)$ also forms a bounded sequence. Since the inclusion $V \subset L_4(\Omega)^n$ is completely continuous, the Arzelá–Ascoli theorem yields the relative compactness of the sequence $\{\widetilde{v}_m\}_{m=1}^\infty$ in the space $C([0,T]; L_4(\Omega)^n)$. So, we can select a subsequence $\{\widetilde{v}_{m_k}\}_{k=1}^\infty$ which is convergent in $C([0,T], L_4(\Omega)^n)$ and hence fundamental. Applying estimate (3.10), we obtain that the sequence $\{B(\widetilde{v}_{m_k})\}_{k=1}^\infty = \{B(v_{m_k})\}_{k=1}^\infty$ is also fundamental and therefore convergent in the Banach space C_*. □

Now, let us define the following continuous operators from C^1 into $C_* \times \mathcal{C}$ setting

$$\mathcal{L}(v) = ((\mu_2 A + J)\widetilde{v}\,' + \mu_1 A\widetilde{v}, v_0),\ \text{where } \widetilde{v} = v|_{[0,T]};$$

$$\widetilde{B}(v) = (B(v), 0);$$

and the u.s.c. multivalued operator $\widetilde{\mathfrak{F}}: C^1 \to Kv(C_* \times \mathcal{C})$,

$$\widetilde{\mathfrak{F}}(v) = (\mathfrak{F}(v), \nu).$$

Now the problem of finding weak solutions to (2.1) — (2.8) takes the form of the following operator inclusion

$$(3.11)\qquad \mathcal{L}(v) - \widetilde{B}(v) \in \widetilde{\mathfrak{F}}(v).$$

Let us mention that the boundedness of the operators $(J+\mu_2 A)^{-1}$ and A implies the existence and the continuity of the inverse operator $\mathcal{L}^{-1} : C_* \times \mathcal{C} \to C^1$ (see [3], Ch. 5, Theorem 1.2). Therefore problem (3.11) is equivalent to the existence of a fixed point

$$v \in F(v), \tag{3.12}$$

for the multimap $F = \mathcal{L}^{-1} \circ (\widetilde{\mathfrak{F}} + \widetilde{B}) : C^1 \to Kv(C^1)$.

Lemma 3.5. *Suppose that*

$$l < \|\mathcal{L}\|, \tag{3.13}$$

where l is the Lipshitz constant from condition $(f_2 1)$. Then the multimap F is (k,χ)–condensing on each bounded subset of C^1, where $k = l\,\|\mathcal{L}^{-1}\|$.

Proof. Applying Proposition 1.5 and Lemma 3.4(b) we can see that the multioperator $\widetilde{\mathfrak{F}} + \widetilde{B}$ is (l,χ,χ')–bounded, where χ and χ' are the Hausdorff MNCs in C^1 and $C_* \times \mathcal{C}$ respectively. It remains only to use Proposition 1.3 to obtain the desired result. □

In the sequel we will always suppose that condition (3.13) is fulfilled.

4. EXISTENCE OF A WEAK SOLUTION AND OPTIMIZATION OF A FUNCTIONAL

4.1. **An a priori estimate.** In this section we will obtain an a priori estimate for solutions of a parametrized version of inclusion (3.11). This estimate will be used for the proof of existence of weak solutions to problem (2.1) — (2.8).

Theorem 4.1. *If $v \in C^1$ is a solution of the operator inclusion*

$$\mathcal{L}(v) - \lambda \widetilde{B}(v) \in \lambda \widetilde{\mathfrak{F}}(v). \tag{4.1}$$

for some $\lambda \in [0,1]$ then

$$\|v\|_{C^1} \le K(\nu, \mu_1, \mu_2, T, \mathcal{N}), \tag{4.2}$$

where K is a nonnegative number depending only on above arguments.

Proof. Identifying for simplicity the function v with its restriction to $[0,T]$ we have

$$(\mu_2 A + J)v' + \mu_1 A v - \lambda B(v) = \lambda f \ \text{ for some } \ f \in \mathfrak{F}(v), \tag{4.3}$$

whereas on $[-h, 0]$ the following relation holds

$$v_0 = \lambda \nu. \tag{4.4}$$

For each $s \in [0,T]$, take the functional defined by the left-hand side of (4.3) and apply it to $v(s)$. Then we have

$$(4.5)\quad < (\mu_2\widehat{A}+\widehat{J})v'(s), v(s) > + < \mu_1\widehat{A}v(s), v(s) > - \\ - \lambda < \widehat{B}v(s), v(s) >= \lambda < f(s), v(s) > .$$

Integrating (4.5) from 0 to t and taking into account the equality

$$< \widehat{B}v(s), v(s) >= 0 \text{ for all } s \in [0,T]$$

we obtain

$$(4.6)\quad \int_0^t < (\mu_2\widehat{A}+\widehat{J})v'(s), v(s) > ds + \int_0^t < \mu_1\widehat{A}v(s), v(s) > ds = \\ = \lambda \int_0^t < f(s), v(s) > ds.$$

Using the definitions of the operators $\widehat{A}$ and $\widehat{J}$ and the integrating by parts formula, we have

$$\int_0^t < \mu_2\widehat{A}v'(s), v(s) > ds = \mu_2 \int_0^t \big(v'(s), v(s)\big)_V ds$$
$$= \frac{\mu_2}{2}\int_0^t \frac{d}{ds}\big(v(s), v(s)\big)_V ds = \frac{\mu_2}{2}\int_0^t \frac{d}{ds}(\|v(s)\|_V^2)\, ds$$
$$= \frac{\mu_2}{2}\|v(t)\|_V^2 - \frac{\mu_2\lambda}{2}\|\nu(0)\|_V^2$$

and

$$\int_0^t < \widehat{J}v'(s), v(s) > ds = \int_0^t\int_\Omega < v'(s), v(s) > dx\, ds$$
$$= \frac{1}{2}\int_0^t \frac{d}{ds}\left(\int_\Omega (v(s), v(s))_H\, dx\right) ds = \frac{1}{2}\int_0^t \frac{d}{ds}\big(\|v(s)\|_H^2\big) ds$$
$$= \frac{1}{2}\|v(t)\|_H^2 - \frac{\lambda}{2}\|\nu(0)\|_H^2.$$

Now equation (4.6) may be rewritten in the following form

$$\frac{\mu_2}{2}\|v(t)\|_V^2 + \frac{1}{2}\|v(t)\|_H^2 = \frac{\lambda\mu_2}{2}\|\nu(0)\|_V^2 + \frac{\lambda}{2}\|\nu(0)\|_H^2 -$$
$$- \int_0^t < \mu_1\widehat{A}v(s), v(s) > ds + \lambda\int_0^t < f(s), v(s) > ds$$
$$\le \frac{\mu_2}{2}\|\nu(0)\|_V^2 + \frac{1}{2}\|\nu(0)\|_H^2 +$$
$$+ \left|\int_0^t < \mu_1\widehat{A}v(s), v(s) > ds\right| + \left|\int_0^t < f(s), v(s) > ds\right|.$$

Let us estimate the last two terms in the above relation. We have

$$\left|\int_0^t < \mu_1\widehat{A}v(s), v(s) > ds\right| = \left|\int_0^t \mu_1\big(v(s), v(s)\big)_V ds\right|$$

$$= \left| \int_0^t \mu_1 \|v(s)\|_V^2 ds \right| \leq |\mu_1| \int_0^t \|v(s)\|_V^2 ds$$

and

$$\left| \int_0^t < f(s), v(s) > ds \right| \leq \int_0^t \|f(s)\|_{V^*} \|v(s)\|_V ds \leq$$

$$\leq \frac{1}{2} \int_0^t \|f(s)\|_{V^*}^2 ds + \frac{1}{2} \int_0^t \|v(s)\|_V^2 ds \leq$$

$$\leq \frac{T}{2} \|f\|_{C_*}^2 + \frac{1}{2} \int_0^t \|v(s)\|_V^2 ds \leq \frac{T\mathcal{N}^2}{2} + \frac{1}{2} \int_0^t \|v(s)\|_V^2 ds$$

(see Remark 2.1).

So we obtain the following estimate

$$\frac{\mu_2}{2} \|v(t)\|_V^2 \leq \frac{T\mathcal{N}^2}{2} + \frac{\mu_2}{2} \|\nu(0)\|_V^2 + \frac{1}{2} \|\nu(0)\|_H^2 + \frac{1}{2} \int_0^t \|v(s)\|_V^2 ds$$

from which we conclude that

$$\|v(t)\|_V^2 \leq \frac{1}{\mu_2} \big(T\mathcal{N}^2 + \|\nu(0)\|_V^2 + \mu_2 \|\nu(0)\|_H^2 \big) + \frac{1}{\mu_2} \int_0^t \|v(s)\|_V^2 ds. \tag{4.7}$$

Applying to (4.7) the Gronwall inequality we have

$$\|v(t)\|_V^2 \leq \frac{1}{\mu_2} \big(T\mathcal{N}^2 + \|\nu(0)\|_V^2 + \mu_2 \|\nu(0)\|_H^2 \big) \, e^{\frac{T}{\mu_2}}$$

and therefore

$$\|v\|_C \leq \sqrt{\frac{1}{\mu_2} \big(T\mathcal{N}^2 + \|\nu(0)\|_V^2 + \mu_2 \|\nu(0)\|_H^2 \big) \, e^{\frac{T}{\mu_2}}}. \tag{4.8}$$

To obtain the estimate for $\|v'\|_C$ let us apply the operator $(\mu_2 A + J)^{-1}$ to both sides of equation (4.3). We have

$$v' = (\mu_2 A + J)^{-1} (\lambda B(v) - \mu_1 A v + \lambda f)$$

and hence

$$\|v'\|_C \leq \|(\mu_2 A + J)^{-1}\| \big(\|B(v)\|_{C_*} + |\mu_1| \, \|Av\|_{C_*} + \mathcal{N} \big). \tag{4.9}$$

Now let us mention that the continuous inclusion $V \subset L_4(\Omega)^n$ implies the relation

$$\|\varphi\|_{L_4(\Omega)^n} \leq \alpha \|\varphi\|_V \ \text{ for all } \ \varphi \in V$$

for some constant $\alpha > 0$. Then inequality (3.10) obviously implies the estimate

$$\|B(v)\|_{C_*} \leq \alpha c_2 \|v\|_C^2. \tag{4.10}$$

Applying also relation (3.3), we obtain from (4.9) and (4.10):

$$\|v'\|_C \leq \|(\mu_2 A + J)^{-1}\| \big(\alpha c_2 \|v\|_C^2 + |\mu_1| \, \|v\|_C + \mathcal{N} \big). \tag{4.11}$$

Now we deduce the desired result from (4.8) and (4.11). □

4.2. **PROOF OF THE MAIN RESULT.** The above a priori estimate allows to apply the topological degree method to obtain our main result, Theorem 2.4.

As we know, the existence of a weak solution to (2.1) — (2.8) can be reduced to fixed point problem (3.12). From Theorem 4.1 it follows that there exists a closed ball $\overline{B}_R \subset C^1$ centered at the origin of the radius $R > 0$ such that

$$v \notin \lambda F(v) \qquad \text{for all } (v, \lambda) \in \partial B_R \times [0, 1].$$

Applying Lemma 3.5 and the properties of the topological degree (see Section 1) we have

$$deg(i - F, \overline{B}_R) = deg(i, \overline{B}_R) = 1$$

yielding the existence of a fixed point of F in B_R.

So we see that the set Σ of all weak solutions to problem (2.1) — (2.8) is nonempty. Moreover, since the multimap F is u.s.c. and condensing, the set of all $v \in C^1$ such that $(v, f) \in \Sigma$ is compact (see [4], Proposition 3.5.1). The feedback relation

$$f \in \mathfrak{F}(v)$$

for each $(v, f) \in \Sigma$ implies that the corresponding set of $f \in C_*$ is also compact (see Remark 2.2 and Proposition 1.2). And now we conclude that the quality functional j admits a minimizer on the compact set Σ.

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