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A BOUNDING FUNCTIONS APPROACH TO MULTIVALUED BOUNDARY VALUE PROBLEMS

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ABSTRACT. The solvability of Floquet boundary value problems is investigated for upper-Carathéodory differential inclusions by means of strictly localized C^2-bounding functions. The existence of an entirely bounded solution is obtained in a sequential way. Our criteria can be regarded as a multivalued extension of recent results of Mawhin and Thompson concerning periodic and bounded solutions of Carathéodory differential equations. A simple illustrating example is supplied.

Keywords and phrases: Floquet boundary value problems, upper-Carathéodory differential inclusions, bounding functions, bounded solutions.

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1. INTRODUCTION

We consider the multivalued Floquet boundary value problem (b.v.p.)

$$\begin{cases} x' + A(t)x & \in F(t,x), \quad \text{for a.a. } t \in [a,b], \\ x(b) & = Mx(a). \end{cases} \tag{1}$$

and throughout the paper we assume that the following conditions are satisfied:

(a) $A : [a,b] \to \mathbb{R}^{n\times n}$ is a matrix-valued measurable function such that $|A(t)| \leq \gamma(t)$, for all $t \in [a,b]$, with some integrable function $\gamma : [a,b] \to [0,+\infty)$;

(b) $F : [a,b] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory (u-Carathéodory) set-valued map (see Definition 1 below);

(c) M is a regular $n \times n$ matrix.

Definition 1. Given a real interval J and a set $X \subseteq \mathbb{R}^m$, we say that $F : J\times X \multimap \mathbb{R}^p$, i.e. $F : J \times X \to 2^{\mathbb{R}^p} \setminus \{\emptyset\}$, is an *u-Carathéodory map*, provided

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(i) the values $F(t,x)$ are nonempty, compact and convex sets for every $(t,x) \in J \times X$,

(ii) $F(\cdot,x)$ is measurable, for each $x \in X$,

(iii) $F(t,\cdot)$ is upper semicontinuous (u.s.c.), for almost all (a.a.) $t \in J$,

(iv) $|w| \le r(t)(1+|x|)$, for every $(t,x) \in J \times X$, $w \in F(t,x)$, where $r \in L^1_{loc}(J)$.

In [4], we proposed a method for the investigation of problem (1) which consisted of several steps. We associated to (1) a one-parameter family of linearized problems. Furtermore, we considered a bounded subset $K \subseteq \mathbb{R}^n$ and proved that all linearized problems have no solutions tangent to ∂K. Finally, we employed a continuation principle developed by the first author, jointly with Gabor and Górniewicz (see [2] and [3]).

Let us note that our technique applies to every Floquet problem of type (1), not only for the special case $M = I$ related to periodic solutions. Hence, it enables us to solve problem (1) for a wide class of right-hand sides F (see [2]–[5]).

A delicate point in this investigation is to check the transversality condition required on ∂K, for all linearized problems. We overcome this obstruction when assuming that K is a bound set. The theory of bound sets was introduced by Gaines and Mawhin (see [9]) in the single valued case. In [4], [5], we generalized it for differential inclusions.

We recall that a bound set approach consists of application of a family of Lyapunov-like functions, called in this context bounding functions. As indicated in [13], if F is a Carathéodory map and locally Lipschitzian bounding functions are applied, then transversality conditions should be satisfied all over a neighborhood of ∂K. A typical result of this type was also our Theorem 3.2 in [5] which is slightly modified, according to our needs, as Proposition 2 below.

However, when a more regular bounding function is applied, then an approximation argument of Scorza–Dragoni type (see Proposition 1) allows us to localize the transversality conditions again on ∂K, as in the case of a globally u.s.c. F in [4].

Mawhin and Thompson (see [12]) proposed this technique for the study of periodic solutions in the single valued case. Our aim here is to extended their approch to multivalued setting for Floquet b.v.p. of type (1).

The following result completes the investigations in [4] and [5] concerning b.v.p. (1), in a finite dimensional state space. Due to higher regularity of applied bounding functions, it especially improves the analogous statements in [5].

Theorem 1. *Assume that*

1) *the associated to* (1) *homogenous problem*

$$\begin{cases} x' + A(t)x = 0, & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a), \end{cases}$$

has only the trivial solution;

2) *there exists an u-Carathéodory mapping* $G : [a,b] \times \mathbb{R}^{2n} \times [0,1] \multimap \mathbb{R}^n$ *such that, for all* $t \in [a,b]$, $x, y \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$, *it holds*

$$G(t,x,y,0) = G_0(t,x), \qquad G(t,y,y,1) \subset F(t,y);$$
$$|w| \le s(t)(1 + |x| + |y| + \lambda),$$

with $w \in G(t,x,y,\lambda)$ *and* $s \in L^1([a,b])$*;*

3) *there exist a non-empty, open and bounded set* $K \subset \mathbb{R}^n$, *whose closure is a retract of* $\mathbb{R}^n$, *and whose boundary* ∂K *satisfies* $M\partial K = \partial K$, *and a* C^2*-function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that* $V(x) \le 0$ *on* $\overline{K}$, $V(x) = 0$, $\nabla V(x) \ne 0$ *on* ∂K, *and*

$$< \nabla V(x), w > \le 0, \tag{2}$$

for all $t \in (a,b]$, $x \in \partial K$, $y \in \overline{K}$, $\lambda \in [0,1]$, *and* $w \in G(t,x,y,\lambda) - A(t)x$, *where* $< \cdot, \cdot >$ *denotes the inner product;*

4) $G(t,\cdot,y,\lambda)$ *is Lipschitzian with a sufficiently small Lipschitz constant* L, *for each* $(t,y,\lambda) \in [a,b] \times \overline{K} \times [0,1]$*;*

5) *for each solution* $x(\cdot)$ *of problem*

$$\begin{cases} x' + A(t)x \in G_0(t,x), & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a) \end{cases} \tag{3}$$

it holds $x(t) \in K$ *for all* $t \in [a,b]$.

Then (1) *admits a solution.*

The proof of this theorem is performed in Section 3. It is based on the usage of a sequence of approximating problems involving strict transversality conditions. These approximating problems are solvable according to Proposition 2, and a standard limiting argument is then applied to conclude the proof. It is clear from the proof that every solution $x(\cdot)$ obtained by means of Theorem 1, satisfies $x(t) \in \overline{K}$, for all $t \in [a,b]$.

As far as we know, existence results for problem (1) are rare and they mainly concern periodic solutions. We refer to [2], [5], [10] and [16] for a wide list of related references; for an updated list of references, see also very recent papers [1], [11], [14] and [15].

In Section 4, combining Theorem 1 with a classical sequential approach, we still prove the existence of entirely bounded solutions of inclusion (1) (see Theorem 2). We conclude with an illustrating example.

Given $X \subseteq \mathbb{R}^n$ and $\delta > 0$, we put

$$B_X^\delta := \bigcup_{x \in X} B_x^\delta$$

where B_x^δ denotes the ball centered in x and of radius δ.

As usual, the set $C([a,b], X)$ is the Banach space of all continuous functions $x : [a,b] \to X$ endowed with the sup-norm.

2. PRELIMINARY RESULTS

For our purposes, we need the following Scorza–Dragoni type result for multivalued maps (see [8, Proposition 5.1]).

Proposition 1. *Let $F : [a,b] \times X \multimap \mathbb{R}^n$ be an u-Carathéodory map such that $X \subset \mathbb{R}^n$ is compact. Then there exists a multivalued mapping $F_0 : [a,b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t,x) \subset F(t,x)$, for all $(t,x) \in [a,b] \times X$, having the following properties:*

(i) *if $u, v : [a,b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t, u(t))$, on $[a,b]$, then $v(t) \in F_0(t, u(t))$, a.e. on $[a,b]$;*
(ii) *for every $\epsilon > 0$, there exists a closed $I_\epsilon \subset [a,b]$ such that $\lambda([a,b] \setminus I_\epsilon) < \epsilon$, $F_0(t,x) \neq \emptyset$, and it is u.s.c. on $I_\epsilon \times X$.*

We now state an existence result for problem (1) which is a slight modification of the main existence result in [5]. It requires a sharp transversality condition (see (4) below) and will be a tool for our present investigation.

Proposition 2. *Assume that*

1) *the associated homogeneous problem*

$$\begin{cases} x' + A(t)x = 0, & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a) \end{cases}$$

has only the trivial solution;

2) *there exists an u-Carathéodory mapping $G : [a,b] \times \mathbb{R}^{2n} \times [0,1] \multimap \mathbb{R}^n$ such that, for all $t \in [a,b]$, $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, it holds*

$$G(t,x,y,0) = G_0(t,x), \qquad G(t,y,y,1) \subset F(t,y);$$
$$|w| \le s(t)(1 + |x| + |y| + \lambda),$$

where $w \in G(t,x,y,\lambda)$ and $s \in L^1([a,b])$;

3) *there exists a non-empty, open and bounded subset K of $\mathbb{R}^n$, whose closure is a retract of $\mathbb{R}^n$, satisfying $M\partial K = \partial K$, and a C^1- function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) \le 0$ on $\overline{K}$, $V(x) = 0$ on ∂K, and*

$$\langle \nabla V(x), w \rangle < 0, \tag{4}$$

for all $t \in (a,b]$, $x \in \overline{K} \cap B^{\epsilon}_{\partial K}$, $y \in \overline{K}$, $\lambda \in (0,1]$ *and* $w \in G(t,x,y,\lambda) - A(t)x$, *with* $\epsilon > 0$*;*

4) *for each* $(t,y,\lambda) \in [a,b] \times \overline{K} \times [0,1]$, $G(t,\cdot,y,\lambda)$ *is Lipschitzian with a Lipschitz function* $L(t) = L + l(t)$ *such that* L *as well as* $\int_a^b l(t)dt$ *are sufficiently small;*
5) *the set of solutions of*

$$\begin{cases} x' + A(t)x \in G_0(t,x), & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a) \end{cases}$$

is a subset of $C([a,b],K)$.

Then (1) *admits a solution.*

Sketch of the proof. The result can be obtained as a special case of [5, Theorem 3.2] (see also [3, Theorem (8.77)]). Condition 5) guarantees that problem (3) has no solutions on the boundary of $C([a,b],\overline{K})$. Property (iv) in Definition 1, jointly with Gronwall's lemma, assure that all solutions $x(\cdot)$ of

$$x'(t) + A(t)x(t) \in G(t,x(t),q(t),\lambda) \tag{5}$$

are equi-bounded, for every $q \in C([a,b],\overline{K})$ and $\lambda \in [0,1]$. Consequently, there exists an integrable $c : [a,b] \to [0,+\infty)$ such that $|w| \leq c(t)$, a.e. in $[a,b]$, for all $w \in G(t,x(t),q(t),\lambda)$, $q \in C([a,b],\overline{K})$, $\lambda \in [0,1]$, and solution $x(\cdot)$ of (5). The proof is the same as in [5], apart from two slight modifications. The first one is related to the fact that $A(\cdot)$ can be taken integrable, instead of to be continuous. The second modification consists in replacing a sufficiently small Lipschitz constant of the function G w.r.t. x by an L^1-Lipschitz function $L(\cdot)$ such that $\int_a^b L(t)dt$ is sufficiently small. These two changes are standard and they do not affect the conclusion concerning the R_δ-structure of the solution set, when following step by step the proof of the main result in [6].

Remark 1. In [7] (see also [3, p. 283]), the same Lipschitz-type assumption was employed, in order to guarantee the R_δ-structure of the solution set. It was explicitly expressed, but only in the single valued case.

3. **PROOF OF THEOREM 1**

Proof. According to condition 3), $\nabla V(x) \neq 0$, for all $x \in \partial K$. Moreover, ∂K is compact. Thus, we get that there exist $\delta, \gamma > 0$ such that $\nabla V(x) \neq 0$, for all $x \in B^{\delta}_{\partial K}$, and $|\nabla V(x)| \geq \gamma$, for each $x \in \partial K$. Since $\overline{K}$ is also compact, there exists $\mu \in C^1(\mathbb{R}^N,[0,1])$ such that $\mu \equiv 1$ on $B^{\frac{\delta}{2}}_{\partial K}$, and $\mu \equiv 0$ on $\mathbb{R}^n \setminus B^{\delta}_{\partial K}$.

Consider an open bounded set $K_0 \subset \mathbb{R}^n$ such that $\overline{K} \subset K_0$. Since G is an u-Carathéodory mapping and $A(\cdot)$ is a measurable operator satisfying assumption (a), we can apply a Scorza–Dragoni type result (see e.g. Proposition 1 and [12, Theorem 2.3]).

We obtain the existence of a monotone decreasing sequence $\{\theta_m\}_m$ of subsets of $[a,b]$ and a measurable map $\overline{G} : [a,b] \times \overline{K_0} \times \overline{K_0} \times [0,1] \multimap \mathbb{R}^n$ such that, for every $m \in \mathbb{N}$, $[a,b] \setminus \theta_m$ is compact, $\lambda(\theta_m) < \frac{1}{m}$, $\overline{G}(t,x,y,\lambda) \subset G(t,x,y,\lambda)$, $\overline{G}$ is u.s.c. in $([a,b] \setminus \theta_m) \times \overline{K_0} \times \overline{K_0} \times [0,1]$ and A is continuous in $([a,b] \setminus \theta_m)$. Obviously, $\cap_{m=1}^{\infty} \theta_m$ has zero Lebesgue measure and $\lim_{m \to \infty} \chi_{\theta_m}(t) = 0$, for every $t \notin \cap_{m=1}^{\infty} \theta_m$. Thus, $\overline{G}$ is an u-Carathéodory multivalued mapping.

Now, consider

$$
\begin{aligned}
\hat{G} : [a,b] \times \mathbb{R}^{2n} \times [0,1] &\multimap \mathbb{R}^n \\
(t,x,y,\lambda) &\multimap \begin{cases} \overline{G}(t,x,y,\lambda) & \text{if } (x,y) \in K_0 \times K_0 \\ G(t,x,y,\lambda), & \text{otherwise.} \end{cases}
\end{aligned} \tag{6}
$$

Since K_0 is open and $\overline{G}(t,x,y,\lambda) \subset G(t,x,y,\lambda)$, it follows that $\hat{G}$ is also an u-Carathéodory mapping.

Define, for each $m \in \mathbb{N}$ and $(t,x,y,\lambda) \in [a,b] \times \mathbb{R}^{2n} \times [0,1]$,

$$F_m(t,x) := F(t,x) - \mu(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|},$$

and

$$G_m(t,x,y,\lambda) := \hat{G}(t,x,y,\lambda) - \mu(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|},$$

where $p(t) = 2s(t)(1+\sigma) + \gamma(t)\sigma$ with $\sigma := \max_{x \in \overline{K}} |x|$ and $s(t)$, as in condition 2).

By definition, for each $(t,x) \in [a,b] \times \mathbb{R}^n$ and every $w_m \in F_m(t,x)$, there exists $w_F \in F(t,x)$ such that

$$w_m = w_F - \mu(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|}.$$

Hence, according to the definition of μ, $w_m = w_F$, when $x \in \mathbb{R}^n \setminus B^{\delta}_{\partial K}$, while $\nabla V(x) \neq 0$, when $x \in B^{\delta}_{\partial K}$. Moreover, $|w_m| \leq |w_F| + p(t) + 1 \leq r(t)(1+|x|) + p(t) + 1$. Therefore, F_m is a well-defined u-Carathéodory map on $[a,b] \times \mathbb{R}^n$. Similarly, for each $(t,x,y,\lambda) \in [a,b] \times \mathbb{R}^{2n} \times [0,1]$ and every $w_m \in \hat{G}_m(t,x,y,\lambda)$, $|w_m| \leq s(t)(1+|x|+|y|+\lambda) + p(t) + 1$. Thus, G_m is a also well-defined u-Carathéodory map.

Let us prove that there exists m_0 such that, for all $m \geq m_0$, the b.v.p.

$$
\begin{cases} x' + A(t)x \in F_m(t,x), & t \in [a,b], \\ x(b) = Mx(a), \end{cases} \tag{7}
$$

satisfies the assumptions of Proposition 2. Assumption 1) of Proposition 2 is trivially satisfied, for all $m \in \mathbb{N}$. Now, we show that G_m satisfies assumption 2) of Proposition 2, for all $m \in \mathbb{N}$. In fact, because of $\hat{G}(t,x,y,\lambda) \subset G(t,x,y,\lambda)$, we have

$$G_m(t,y,y,1) \subset G(t,y,y,1) - \mu(y) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(y)}{|\nabla V(y)|} \subset F_m(t,y),$$

for all $(t, y) \in [a, b] \times \mathbb{R}^n$. Moreover, for all $(t, x, y) \in [a, b] \times \mathbb{R}^{2n}$, we obtain

$$G_m(t, x, y, 0) \subset G_0(t, x) - \mu(x)\left(p(t)\chi_{\theta_m}(t) + \frac{1}{m}\right)\frac{\nabla V(y)}{|\nabla V(y)|}.$$

Now, let us investigate assumption 3) in Proposition 2. For every $(t, x, y, \lambda) \in \theta_m \times (B_{\partial K}^{\frac{\delta}{2}} \cap \overline{K}) \times \overline{K} \times [0, 1]$ and $w_m \in G_m(t, x, y, \lambda) - A(t)x$, we obtain

$$\begin{aligned} \langle \nabla V(x), w_m \rangle &= \langle \nabla V(x), w_G - A(t)x \rangle - \left(p(t) + \frac{1}{m}\right)|\nabla V(x)| \\ &\leq \left[|w_G| + |A(t)x| - \left(p(t) + \frac{1}{m}\right)\right]|\nabla V(x)| \\ &\leq \Bigg[s(t)(1 + |x| + |y| + \lambda) + \gamma(t)|x| \\ &\quad -\left(p(t) + \frac{1}{m}\right)\Bigg]|\nabla V(x)| \\ &\leq -\frac{1}{m}|\nabla V(x)| \\ &< 0, \end{aligned}$$

because $\mu \equiv 1$ in $B_{\partial K}^{\frac{\delta}{2}} \cap \overline{K}$.

Let us consider $t \in [a, b] \setminus \theta_m$. According to assumption 3), for every $x \in \partial K$, $y \in \overline{K}$, $\lambda \in [0, 1]$ and $w_m \in G_m(t, x, y, \lambda) - A(t)x$,

$$\begin{aligned} \langle \nabla V(x), w_m \rangle &= \langle \nabla V(x), w_G - A(t)x \rangle - \frac{1}{m}|\nabla V(x)| \\ &\leq -\frac{1}{m}|\nabla V(x)| \\ &\leq -\frac{\gamma}{m}, \end{aligned}$$

because $|\nabla V(x)| \geq \gamma$, for each $x \in \partial K$. Since $\hat{G} \equiv \overline{G}$ which is u.s.c. in $([a, b] \setminus \theta_m) \times \overline{K} \times \overline{K} \times [0, 1]$, $A \in C([a, b] \setminus \theta_m)$ and $V \in C^2(\mathbb{R}^n)$, it follows that $(t, x) \multimap \langle \nabla V(x), G_m(t, x, y, \lambda) - A(t)x \rangle$ is u.s.c. on the compact set $([a, b] \setminus \theta_m) \times \overline{K} \times \overline{K} \times [0, 1]$. Hence, the existence of $\sigma_m > 0$ is implied such that

$$\langle \nabla V(x), w_m \rangle < 0,$$

for every $t \in [a, b] \setminus \theta_m$, $x \in B_{\partial K}^{\sigma_m} \cap \overline{K}$, $y \in \overline{K}$, $\lambda \in [0, 1]$, $w_m \in G_m(t, x, y, \lambda) - A(t)x$, and assumption 3) of Proposition 2 follows, when taking $\epsilon_m = \min(\sigma_m, \frac{\delta}{2})$.

Let us also notice that, since μ and V are respectively of class C^1 and C^2, denoting by P the Lipschitz constant of $\mu\frac{\nabla V}{|\nabla V|}$, $G_m(t, \cdot, y, \lambda)$ is Lipschitzian with Lipschitz function $L + l(t)$, where $l(t) := (p(t)\chi_{\theta_m}(t) + \frac{1}{m})P$. Since $p(\cdot) \in L^1[a, b]$ and $\lambda(\theta_m) < \frac{1}{m}$, it is possible to find $\overline{m}$ such that, for all $m \geq \overline{m}$, $\int_a^b l(t)dt$ is sufficiently small, and condition 4) implies assumption 4) of Proposition 2.

Finally, recalling that $\hat{G}(t,x,y,0) \subset G(t,x,y,0) = G_0(t,x)$, for each $(t,x,y) \in [a,b] \times R^n \times R^n$, consider a solution $x(\cdot)$ of the b.v.p.

$$\begin{cases} x' + A(t)x \in G_0(t,x) - \mu(x)\big(p(t)\chi_{\theta_m}(t) + \frac{1}{m}\big)\frac{\nabla V(x)}{|\nabla V(x)|}, \\ x(b) = Mx(a), \qquad t \in [a,b]. \end{cases} \tag{8}$$

Only two cases can occur: $x(t) \in \mathbb{R}^n \setminus B^\delta_{\partial K}$, for all $t \in [a,b]$ or there exists $t_0 \in [a,b]$ such that $x(t_0) \in B^\delta_{\partial K}$. According to condition 5), in the first case, $x(\cdot)$ is a solution of (3), because $\mu(x(t)) = 0$, for all t. Hence, $|x(t)| \le \sigma$ for all $t \in [a,b]$. We show that also in the second case the solutions $x(\cdot)$ are equi-bounded. In fact, first assuming $t \in [t_0, b]$, integrating the inclusion in (8) and passing to the norm, we get

$$\begin{aligned} |x(t)| \;&\le\; |x(t_0)| + \int_{t_0}^t \gamma(\tau)|x(\tau)|\,d\tau + \int_{t_0}^t s(\tau)(1+|x(\tau)|)\,d\tau \\ &+\; \int_{t_0}^t (p(\tau)+1) \;\le \alpha + \int_a^t (\gamma(\tau)+s(\tau))|x(\tau)|\,d\tau, \end{aligned}$$

where $\alpha := \sigma + \delta + \int_a^b (s(\tau)+p(\tau)+1)\,d\tau$. By Gronwall's lemma, we obtain

$$|x(t)| \le \alpha \mathrm{e}^{\int_a^b (\gamma(\tau)+s(\tau))\,d\tau}, \qquad t_0 \le t \le b.$$

Notice that previous estimate holds also when $t \in [a, t_0]$, so we have proved that the solutions of (8) are equi-bounded, independently of $m \in \mathbb{N}$. Let us suppose that, for each $m \in \mathbb{N}$, there exist $p_m \ge m$, solution $x_{p_m}(\cdot)$ of (8) and $t_{p_m} \in [a,b]$ such that $x_{p_m}(t_{p_m}) \notin K$. We note that, since G is of u-Carathéodory type (see condition 2)) and $\{x_{p_m}(\cdot)\}_m$ is an equi-bounded sequence, then $\{x'_{p_m}(\cdot)\}_m$ is also equi-bounded in $L^1[a,b]$. Indeed,

$$|x'_{p_m}(t)| \le \gamma(t)|x_{p_m}(t)| + s(t)(1+|x_{p_m}(t)|) + p(t) + 1,$$

for all $t \in [a,b]$ and $m \in \mathbb{N}$. Thus, the Ascoli–Arzelà theorem implies that $\{x_{p_m}(\cdot)\}_m$ has a subsequence, again denoted as the sequence, such that $x_{p_m}(t) \to x(t)$, as $m \to +\infty$, uniformly in $[a,b]$, the function $x(\cdot)$ is absolutely continuous in $[a,b]$, and $x'_{p_m} \rightharpoonup x'$, weakly in $L^1[a,b]$. Notice, moreover, that since $\lim\limits_{m\to\infty} \chi_{\theta_m}(t) = 0$, for every $t \notin \cap_{m=1}^\infty \theta_m$, and $\lambda(\cap_{m=1}^\infty \theta_m) = 0$,

$$\mu(x_{p_m})\Big(p(t)\chi_{\theta_{p_m}} + \frac{1}{p_m}\Big)\frac{\nabla V(x_{p_m}(t))}{|\nabla V(x_{p_m}(t))|} \to 0, \quad \text{a.e. in } [a,b].$$

Consequently, a standard limiting argument (see e.g. [16, page 88]) implies that $x(\cdot)$ is a solution of (3). Hence, according to 5), we obtain that $x(t) \in K$, for all $t \in [a,b]$. On the other hand, $\{t_{p_m}\}_m$ has a subsequence which converges to some $\bar{t} \in [a,b]$. This leads to a contradiction with $x_{p_m}(t_{p_m}) \notin K$, for all m, because K is open. So, we can conclude that there exists $\overline{\overline{m}}$ such that, for all $m \ge \overline{\overline{m}}$ and solution $x_m(\cdot)$ of (8), it holds $x_m(t) \in K$, for all $t \in [a,b]$.

Hence, we can apply Proposition 2, obtaining, for every $m \geq \max\{\overline{m}, \overline{\overline{m}}\}$, a solution $x_m(\cdot)$ of (7) such that $x_m(t) \in \overline{K}$, for each $t \in [a,b]$. Passing to a subsequence, again denoted as the sequence, also in this case we can show the existence of an absolutely continuous function $x : [a,b] \to \overline{K}$ such that $x_m(\cdot) \to x(\cdot)$ as $m \to +\infty$, uniformly in $[a,b]$ and $x'_m \rightharpoonup x'$ weakly in $L^1[a,b]$. The same limiting argument mentioned above implies that $x(\cdot)$ is a solution of problem (1) which completes the proof. $\square$

Remark 2. A typical case occurs when $G(t,x,y,\lambda) = \lambda F(t,y) + (1-\lambda)G_0(t,x)$, where G_0 is an u-Carathéodory multivalued mapping. If, in particular, $G_0(t,x) \equiv 0$, then according to condition 1), the set of solutions of (3) is a subset of $C([a,b],K)$ if and only if K is a neighborhood of the origin.

4. BOUNDED SOLUTIONS

Consider the inclusion

$$x' \in F(t,x), \quad \text{for a.a. } t \in \mathbb{R}, \tag{9}$$

where $F : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an u-Carathéodory set-valued map. Combining Theorem 1 with a classical sequential approach, we would like to guarantee the existence of an entirely (i.e. on the whole real line) bounded solution of (9). This result is a generalization of [5, Theorem 4.2], because the strict monotonicity of the bounding function V along the solutions of (9) is now required only at the boundary ∂K of the bound set K.

Theorem 2. *Assume that*

1) *there exists an u-Carathéodory mapping $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \multimap \mathbb{R}^n$ such that, for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^{2n}$ and $\lambda \in [0,1]$, it follows;*

$$\begin{aligned} &G(t,x,y,0) = G_0(t,x), \qquad G(t,y,y,1) \subset F(t,y); \\ &|w| \leq s(t)(1 + |x| + |y| + \lambda), \end{aligned}$$

with $w \in G(t,x,y,\lambda)$ and $s \in L^1_{loc}(\mathbb{R})$;

2) *there exists a bounded, non-empty and open $K \subset \mathbb{R}^n$, symmetric with respect to the origin, whose closure is a retract of $\mathbb{R}^n$, and a function $V : \mathbb{R}^n \to \mathbb{R}$ of class C^2 such that $V(x) \leq 0$ on $\overline{K}$, $V(x) = 0$ and $\nabla V(x) \neq 0$, for every $x \in \partial K$, and*

$$\langle \nabla V(x), w \rangle \leq 0,$$

for all $t \in \mathbb{R}$, $x \in \partial K$, $y \in \overline{K}$, $\lambda \in [0,1]$, $w \in G(t,x,y,\lambda)$;

3) *$G(t,\cdot,y,\lambda)$ is Lipschitzian with a sufficiently small Lipschitz constant L, for every $(t,y,\lambda) \in \mathbb{R} \times \overline{K} \times [0,1]$;*

4) *for each* $m \in \mathbb{N}$, *the set of solutions of*

$$\begin{cases} x' \in G_0(t,x), & \text{for a.a. } t \in [a,b], \\ x(m) = -x(-m) \end{cases} \tag{10}$$

is a subset of $C([-m,m],K)$.

Then (9) *admits a bounded solution.*

Proof. Given $m \in \mathbb{N}$, let us consider problem (1) with $[a,b] = [-m,m]$, $M = -I$ and $A \equiv 0$, i.e. the anti-periodic boundary value problem on $[-m,m]$:

$$\begin{cases} x' \in F(t,x), & \text{for a.a. } t \in [-m,m], \\ x(m) = -x(-m), \end{cases} \tag{11}$$

whose associated homogeneous problem has only the trivial solution. In this case, the invariance of ∂K with respect to M is equivalent to the symmetry of ∂K with respect to the origin. Thus, all the assumptions of Theorem 1 are satisfied, and we get, for each $m \in \mathbb{N}$, the existence of solutions $x_m(\cdot)$ of (11) such that $x_m(t) \in \overline{K}$, for all $t \in [-m,m]$. The conclusion follows from Lemma 4.1 in [5] (cf. Proposition III.1.37 in [3]). □

Example 1. Consider the differential inclusion

$$x' \in F_1(t,x) + F_2(t,x), \tag{12}$$

where $F_1, F_2 : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ are u-Carathéodory maps and $F_1(t,\cdot)$ is Lipschitzian, with a sufficiently small Lipschitz constant, for all $t \in \mathbb{R}$. Assume, furthermore, the existence of a positive constant R such that $\langle x, w \rangle \le 0$, for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, with $|x| = R$ and $|y| \le R$, and $w \in F_1(t,x) + F_2(t,y)$.

Take the u-Carathéodory map $G(t,x,y,\lambda) := \lambda(F_1(t,x) + F_2(t,y))$, and put $K = B_0^R$. Since K is a neighborhood of the origin and $G_0 \equiv 0$, the set of solutions of (10) is, for all m, a subset of $C([-m,m],K)$ (see Remark 2). Furtermore, since all assumptions of Theorem 2 can be satisfied by means of C^2-function $V(x) = |x|^2 - R^2$, the inclusion (12) admits a bounded solution.

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